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Transport equations with disparate advection fields. Application to the gyrokinetic models in plasma physics

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ABSTRACT

The subject matter of this paper concerns the asymptotic regimes for transport equations with advection fields having components of very disparate orders of magnitude. The main purpose is to derive the limit models: we justify rigorously the convergence towards these limit models and we investigate the well-posedness of them. Such asymptotic analysis arises in the magnetic confinement context, where charged particles move under the action of strong magnetic fields. In these situations we distinguish between a slow motion driven by the electric field and a fast motion around the magnetic lines.

1. Introduction

In this work we focus on linear transport problems, where a part of the transport operator is highly penalized

$$\begin{cases} \partial_t u^\varepsilon + a(t,y)\cdot\nabla_y u^\varepsilon + \dfrac{b(y)}{\varepsilon}\cdot\nabla_y u^\varepsilon = 0, & (t,y)\in\mathbb{R}_+\times\mathbb{R}^m, \\ u^\varepsilon(0,y) = u_0^\varepsilon(y), & y\in\mathbb{R}^m. \end{cases} \tag{1}$$

Here a and b are given smooth fields and we also assume that b is divergence free. Clearly we deal with multiple scales: slow advection along a and fast advection along b. Formally, multiplying the transport equation in (1) by ε one gets $b(y)\cdot\nabla_y u^\varepsilon = \mathcal{O}(\varepsilon)$, saying that the variation of u^ε along

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the trajectories of b vanishes as ε goes to zero. Following this observation it may seem reasonable to interpret the asymptotic $\varepsilon \searrow 0$ in (1) as homogenization procedure with respect to the flow of b. More precisely we appeal here to the ergodic theory.

By Hilbert's method we have the formal expansion

$$u^\varepsilon = u + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots \tag{2}$$

and thus, plugging the ansatz (2) in (1) yields the equations

$$\varepsilon^{-1}\colon\ b(y) \cdot \nabla_y u = 0, \tag{3}$$

$$\varepsilon^{0}\colon\ \partial_t u + a(t,y) \cdot \nabla_y u + b(y) \cdot \nabla_y u_1 = 0, \tag{4}$$

$$\varepsilon^{1}\colon\ \partial_t u_1 + a(t,y) \cdot \nabla_y u_1 + b(y) \cdot \nabla_y u_2 = 0, \tag{5}$$

$$\vdots$$

The operator $\mathcal{T} = b(y) \cdot \nabla_y$ will play a crucial role in our analysis: Eq. (3) says that at any time $t \in \mathbb{R}_+$ the leading order term in the expansion (2) belongs to the kernel of $\mathcal{T}$. Certainly this information (which will be interpreted later on as a constraint) is not sufficient for uniquely determining u. The use of (4) is mandatory, despite the coupling with the next term u_1 in the asymptotic expansion (2). Actually, at least in a first step, we do not need all the information in (4), but only some consequence of it, such that, supplemented by the constraint (3), it will allows us to determine u. Since we need to eliminate u_1 in (4), the idea is to project (4) at any time $t \in \mathbb{R}_+$ to the orthogonal complement of the range of $\mathcal{T}$ (which coincides with the kernel of $\mathcal{T}$, since $\operatorname{div}_y b = 0$), for example in $L^2(\mathbb{R}^m)$. Indeed, we will see that this consequence of (4) together with the constraint (3) provide a well-posed limit model for $u = \lim_{\varepsilon \searrow 0} u^\varepsilon$. And the same procedure applies for computing $u_1, u_2, \ldots$. For example, once we have determined u, by (4) we know the image by $\mathcal{T}$ of u_1

$$\mathcal{T} u_1 = -\partial_t u - a(t,y) \cdot \nabla_y u. \tag{6}$$

Projecting now (5) on the orthogonal complement of the range of $\mathcal{T}$ we eliminate u_2 and one gets another equation for u_1, which combined to (6) provides a well-posed problem for u_1. More precisely, if $Y(s;y)$ is the characteristic flow associated to the field b, we denote by $\langle v \rangle$ the average of any function v, let us say in $L^2(\mathbb{R}^m)$, over the flow

$$\langle v \rangle (y) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T v\big(Y(s;y)\big)\,\mathrm{d}s, \quad y \in \mathbb{R}^m.$$

Certainly, the key point which allows us to define the average over the flow is that for any $s \in \mathbb{R}$, the map $y \to Y(s;y)$ is measure preserving. At least formally we have

$$\begin{aligned}
\langle b \cdot \nabla_y u_1 \rangle (y) &= \lim_{T \to +\infty} \frac{1}{T} \int_0^T (b \cdot \nabla_y u_1)\big(Y(s;y)\big)\,\mathrm{d}s \\
&= \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{d}{ds}\big\{u_1\big(Y(s;y)\big)\big\}\,\mathrm{d}s \\
&= \lim_{T \to +\infty} \frac{1}{T} \big\{u_1\big(Y(T;y)\big) - u_1(y)\big\}\,\mathrm{d}s = 0.
\end{aligned}$$

It is easily seen that the average function $\langle v\rangle$ is constant along the flow and satisfies the variational formulation

$$\int_{\mathbb{R}^m} \big(v(y) - \langle v\rangle(y)\big)\varphi(y)\,\mathrm{d}y = 0 \tag{7}$$

for any function φ constant along the flow. In other words, the average $\langle\cdot\rangle$ coincides with the orthogonal projection over the kernel of $\mathcal{T}$. Since the leading order term in (2) is constant along the flow, we have $\langle u\rangle = u$. Therefore applying the average operator in (4) yields the limit model

$$\begin{cases} \partial_t u + \big\langle a(t)\cdot\nabla_y u\big\rangle = 0, \quad b(y)\cdot\nabla_y u = 0, & (t,y)\in\mathbb{R}_+\times\mathbb{R}^m,\\ u(0,y) = u_0(y), & y\in\mathbb{R}^m, \end{cases} \tag{8}$$

where the average $\langle a(t)\cdot\nabla_y u\rangle$ should be understood in the variational sense (7). We develop a weak theory for (8) and justify the convergence of the solutions for (1) towards (8) (see Proposition 3.1). A much difficult task is to identify a strong formulation for (8). The key point here is to determine space derivatives commuting with the average operator. This analysis leads naturally to the notion of fields in involution: a smooth field c is said to be in involution with b iff $[c\cdot\nabla_y, b\cdot\nabla_y] = (c\cdot\nabla_y)(b\cdot\nabla_y) - (b\cdot\nabla_y)(c\cdot\nabla_y) = 0$. It is well known that b, c are in involution iff their corresponding flows Y, Z are commuting

$$Y\big(s; Z(h;\cdot)\big) = Z\big(h; Y(s;\cdot)\big), \quad s,h\in\mathbb{R},$$

and we check that the average operator associated to b is commuting with the directional derivative along any field c in involution with b. For verifying that it is sufficient to observe that the average operator is commuting with the translations along the flow of c and the commutation property between $\langle\cdot\rangle$ and $c\cdot\nabla_y$ follows immediately (see Propositions 2.10, 2.11). When the field a is a linear combination of smooth fields in involution with b

$$a(t,y) = \alpha(t,y)b(y) + \sum_{i=1}^{r}\alpha_i(t,y)b^i(y),$$

$$\big(b^i\cdot\nabla_y\big)(b\cdot\nabla_y) - (b\cdot\nabla_y)\big(b^i\cdot\nabla_y\big) = 0, \quad i\in\{1,\dots,r\},$$

it is shown (cf. Proposition 3.2, Corollary 3.1) that the limit model (8) is equivalent to a linear transport problem which corresponds to the averaged transport operator $\sum_{i=1}^{r}\langle\alpha_i(t)\rangle b^i\cdot\nabla_y$

$$\begin{cases} \partial_t u + \displaystyle\sum_{i=1}^{r}\big\langle\alpha_i(t)\big\rangle b^i\cdot\nabla_y u = 0, & (t,y)\in\mathbb{R}_+\times\mathbb{R}^m,\\ u(0,y) = u_0(y), & y\in\mathbb{R}^m. \end{cases} \tag{9}$$

In this framework we establish a strong convergence result justifying the asymptotic behavior as $\varepsilon\searrow 0$ of the solutions in (1) towards the solution of the transport problem (9).

Theorem 4.1. *Assume that the field a is a linear combination of smooth fields $b^i\in W^{1,\infty}(\mathbb{R}^m)$ in involution with b where α, $(\alpha_i)_{i\in\{1,\dots,r\}}$ are smooth coefficients. Suppose that u_0 and $(u_0^\varepsilon)_{\varepsilon>0}$ are smooth initial conditions such that $b(y)\cdot\nabla_y u_0 = 0$, $\lim_{\varepsilon\searrow 0} u_0^\varepsilon = u_0$ in $L^2(\mathbb{R}^m)$ and let us denote by u, u^ε the solutions of* (9) *and* (1) *respectively. Then we have the strong convergence*

$$\lim_{\varepsilon\searrow 0} u^\varepsilon = u, \quad \text{in } L^\infty\big([0,T]; L^2\big(\mathbb{R}^m\big)\big), \quad \forall T>0.$$

Our paper is organized as follows. In Section 2 we recall some notions of ergodic theory. We introduce the average over a flow associated to a smooth field and we discuss the main properties of this operator. Section 3 is devoted to the study of the limit model. We prove existence, uniqueness and regularity results. The main result (Theorem 4.1) concerning the convergence towards the limit model is justified rigorously in Section 4. In Section 5 we discuss some applications. It turns out that the asymptotic models for strongly magnetized plasmas can be treated by using the general method previously introduced. These limit models follow by applying the main convergence result in Theorem 4.1. Some technical proofs are postponed to Appendices A and B.

2. Ergodic theory and average over a flow

The main tool of our study is the average operator over a flow. In this section we introduce rigorously this notion and investigate its properties. We assume that $b : \mathbb{R}^m \to \mathbb{R}^m$ is a given field satisfying

$$b \in W^{1,\infty}_{\mathrm{loc}}\big(\mathbb{R}^m\big), \tag{10}$$

$$\mathrm{div}_y\, b = 0 \tag{11}$$

and the growth condition

$$\exists C > 0: \quad \big|b(y)\big| \leqslant C\big(1 + |y|\big), \quad y \in \mathbb{R}^m. \tag{12}$$

Under the above hypotheses the characteristic flow $Y = Y(s; y)$ is well defined

$$\frac{dY}{ds} = b\big(Y(s; y)\big), \quad (s, y) \in \mathbb{R} \times \mathbb{R}^m, \tag{13}$$

$$Y(0; y) = y, \quad y \in \mathbb{R}^m, \tag{14}$$

and has the regularity $Y \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^m)$. By (11) we deduce that for any $s \in \mathbb{R}$, the map $y \to Y(s; y)$ is measure preserving

$$\int_{\mathbb{R}^m} \theta\big(Y(s; y)\big)\,\mathrm{d}y = \int_{\mathbb{R}^m} \theta(y)\,\mathrm{d}y, \quad \forall \theta \in L^1\big(\mathbb{R}^m\big).$$

We have the following standard result concerning the kernel of $u \to \mathcal{T}u = \mathrm{div}_y(b(y)u(y))$.

Proposition 2.1. *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$. Then $\mathrm{div}_y(b(y)u(y)) = 0$ in $\mathcal{D}'(\mathbb{R}^m)$ iff for any $s \in \mathbb{R}$ we have $u(Y(s; y)) = u(y)$ for a.a. $y \in \mathbb{R}^m$.*

Remark 2.1. Sometimes we will write $u \in \ker \mathcal{T}$ meaning that u is constant along the characteristics, *i.e.*, $u(Y(s; y)) = u(y)$ for all $s \in \mathbb{R}$ and a.a. $y \in \mathbb{R}^m$.

For any $q \in [1, +\infty]$ we denote by $\mathcal{T}_q$ the linear operator defined by $\mathcal{T}_q u = \mathrm{div}_y(b(y)u(y))$ for any u in the domain

$$\mathrm{D}_q = \big\{u \in L^q\big(\mathbb{R}^m\big):\ \mathrm{div}_y\big(b(y)u(y)\big) \in L^q\big(\mathbb{R}^m\big)\big\}.$$

Thanks to Proposition 2.1 we have for any $q \in [1, +\infty]$

$$\ker \mathcal{T}_q = \big\{u \in L^q\big(\mathbb{R}^m\big):\ u\big(Y(s; y)\big) = u(y),\ s \in \mathbb{R},\ \text{a.e. } y \in \mathbb{R}^m\big\}.$$

For any continuous function $h \in C([a,b]; L^q(\mathbb{R}^m))$, with $q \in [1, +\infty]$, we denote by $\int_a^b h(t)\,\mathrm{d}t \in L^q(\mathbb{R}^m)$ the Riemann integral of the function $t \to h(t) \in L^q(\mathbb{R}^m)$ on the interval $[a,b]$. It is easily seen by the construction of the Riemann integral that for any function $\varphi \in L^{q'}(\mathbb{R}^m)$ (where $1/q + 1/q' = 1$) we have

$$\int_{\mathbb{R}^m} \left(\int_a^b h(t)\,\mathrm{d}t \right)(y)\varphi(y)\,\mathrm{d}y = \int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y)\,\mathrm{d}y \right)\mathrm{d}t \tag{15}$$

implying that

$$\left\| \int_a^b h(t)\,\mathrm{d}t \right\|_{L^q(\mathbb{R}^m)} \leqslant \int_a^b \left\| h(t) \right\|_{L^q(\mathbb{R}^m)}\mathrm{d}t.$$

Moreover, by Fubini theorem we have

$$\int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y)\,\mathrm{d}y \right)\mathrm{d}t = \int_{\mathbb{R}^m} \left(\int_a^b h(t,y)\,\mathrm{d}t \right)\varphi(y)\,\mathrm{d}y$$

which together with (15) yields

$$\left(\int_a^b h(t)\,\mathrm{d}t \right)(y) = \int_a^b h(t,y)\,\mathrm{d}t, \quad \text{a.e. } y \in \mathbb{R}^m.$$

Consider now a function $u \in L^q(\mathbb{R}^m)$. Observing that for any $q \in [1, +\infty)$ the application $s \to u(Y(s;\cdot))$ belongs to $C(\mathbb{R}; L^q(\mathbb{R}^m))$, we deduce that for any $T > 0$ the function $\langle u \rangle_T := \frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s$ is well defined as an element of $L^q(\mathbb{R}^m)$ and $\|\langle u \rangle_T\|_{L^q(\mathbb{R}^m)} \leqslant \|u\|_{L^q(\mathbb{R}^m)}$. Observe that for any function $h \in L^\infty([a,b]; L^\infty(\mathbb{R}^m))$, the map $\varphi \in L^1(\mathbb{R}^m) \to \int_a^b \int_{\mathbb{R}^m} h(t,y)\varphi(y)\,\mathrm{d}y\,\mathrm{d}t$ belongs to $(L^1(\mathbb{R}^m))' = L^\infty(\mathbb{R}^m)$. Therefore there is a unique function in $L^\infty(\mathbb{R}^m)$, denoted $\int_a^b h(t)\,\mathrm{d}t$, such that for any $\varphi \in L^1(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} \left(\int_a^b h(t)\,\mathrm{d}t \right)(y)\varphi(y)\,\mathrm{d}y = \int_a^b \left(\int_{\mathbb{R}^m} h(t,y)\varphi(y)\,\mathrm{d}y \right)\mathrm{d}t.$$

In particular we have

$$\left\| \int_a^b h(t)\,\mathrm{d}t \right\|_{L^\infty(\mathbb{R}^m)} \leqslant \int_a^b \left\| h(t) \right\|_{L^\infty(\mathbb{R}^m)}\mathrm{d}t$$

and as before

$$\left(\int_a^b h(t)\,\mathrm{d}t \right)(y) = \int_a^b h(t,y)\,\mathrm{d}t, \quad \text{a.e. } y \in \mathbb{R}^m.$$

Notice that for any function $u \in L^\infty(\mathbb{R}^m)$, the map $s \to u(Y(s;\cdot))$ belongs to $L^\infty(\mathbb{R}; L^\infty(\mathbb{R}^m))$ and thus we deduce that for any $T > 0$ the function $\langle u\rangle_T := \frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s$ is well defined as an element of $L^\infty(\mathbb{R}^m)$ and $\|\langle u\rangle_T\|_{L^\infty(\mathbb{R}^m)} \leqslant \|u\|_{L^\infty(\mathbb{R}^m)}$.

Obviously, when u belongs to $\ker\mathcal{T}_q$ we have $\langle u\rangle_T = u$ for any $q \in [1,+\infty]$ and $T > 0$. Generally, when $q \in (1,+\infty)$ we prove the weak convergence of $\langle u\rangle_T$ as T goes to $+\infty$ towards some element in $\ker\mathcal{T}_q$. The arguments are standard and can be found in Appendix A.

Proposition 2.2. *Assume that $q \in (1,+\infty)$ and $u \in L^q(\mathbb{R}^m)$. Then there is a unique function $\langle u\rangle \in \ker\mathcal{T}_q$ such that for any $\varphi \in \ker\mathcal{T}_{q'}$ we have*

$$\int_{\mathbb{R}^m} \big(u(y) - \langle u\rangle(y)\big)\varphi(y)\,\mathrm{d}y = 0. \tag{16}$$

Moreover we have the weak convergences in $L^q(\mathbb{R}^m)$

$$\langle u\rangle = \lim_{T\to+\infty}\frac{1}{T}\int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_{-T}^0 u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s$$

and the inequality $\|\langle u\rangle\|_{L^q(\mathbb{R}^m)} \leqslant \|u\|_{L^q(\mathbb{R}^m)}$. In particular the application $u \in L^q(\mathbb{R}^m) \to \langle u\rangle \in L^q(\mathbb{R}^m)$ is linear, continuous and $\|\langle\cdot\rangle\|_{\mathcal{L}(L^q(\mathbb{R}^m),L^q(\mathbb{R}^m))} \leqslant 1$.

It is easily seen that if $m \leqslant u \leqslant M$ then $m \leqslant \langle u\rangle_T \leqslant M$ for any $T > 0$. In particular the average operator preserves the order of $\mathbb{R}$.

Corollary 2.1. *Assume that $q \in (1,+\infty)$ and $u \in L^q(\mathbb{R}^m)$. Let us denote by $\langle u\rangle \in L^q(\mathbb{R}^m)$ the function constructed in Proposition 2.2.*

a) *If $u \geqslant m$ for some real constant m then $\langle u\rangle \geqslant m$.*
b) *If $u \leqslant M$ for some real constant M then $\langle u\rangle \leqslant M$.*

We can prove that the operator $\langle\cdot\rangle$ is local with respect to the trajectories.

Corollary 2.2. *Let $A \subset \mathbb{R}^m$ be an invariant set under the flow Y (i.e., $Y(s;A) \subset A$ for any $s \in \mathbb{R}$). Then for any $u \in L^q(\mathbb{R}^m)$ with $q \in (1,+\infty)$ we have $\langle \mathbf{1}_A u\rangle = \mathbf{1}_A\langle u\rangle$. In particular if $u_1, u_2 \in L^q(\mathbb{R}^m)$ satisfy $u_1 = u_2$ on A, then $\langle u_1\rangle = \langle u_2\rangle$ on A.*

Proof. For any $\varphi \in \ker\mathcal{T}_{q'}$ we have $\int_{\mathbb{R}^m}(u - \langle u\rangle)\varphi\,\mathrm{d}y = 0$. Since A is invariant under the flow, the function $\mathbf{1}_A\varphi$ belongs to $\ker\mathcal{T}_{q'}$ and thus $\int_{\mathbb{R}^m}(u - \langle u\rangle)\mathbf{1}_A\varphi\,\mathrm{d}y = 0$ which says that $\langle\mathbf{1}_A u\rangle = \mathbf{1}_A\langle u\rangle$. If $u_1, u_2 \in L^q(\mathbb{R}^m)$ coincide on A then $\mathbf{1}_A(u_1 - u_2) = 0$. Consequently we have $\mathbf{1}_A\langle u_1 - u_2\rangle = \langle\mathbf{1}_A(u_1 - u_2)\rangle = 0$ saying that $\langle u_1\rangle = \langle u_2\rangle$ on A. $\square$

During our analysis we will use the average operator in different settings $L^q(\mathbb{R}^m)$, $1 < q < +\infty$. A natural question is what happens for functions $u \in L^{q_1}(\mathbb{R}^m) \cap L^{q_2}(\mathbb{R}^m)$; it is true that the averages coincide? The answer to this question is affirmative.

Corollary 2.3. *Assume that $1 < q_1 < q_2 < +\infty$ and $u \in L^{q_1}(\mathbb{R}^m) \cap L^{q_2}(\mathbb{R}^m)$. We denote by $\langle u\rangle^{(q)}$ the function of $L^q(\mathbb{R}^m)$ constructed in Proposition 2.2 for $q \in \{q_1, q_2\}$. Then we have $\langle u\rangle^{(q_1)} = \langle u\rangle^{(q_2)} \in \ker\mathcal{T}_{q_1} \cap \ker\mathcal{T}_{q_2}$.*

Proof. For any $T > 0$ and $\varphi \in C_c(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m} \left(\frac{1}{T} \int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s \right)(y)\varphi(y)\,\mathrm{d}y = \frac{1}{T} \int_0^T \left(\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\varphi(y)\,\mathrm{d}y \right) \mathrm{d}s,$$

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \langle u\rangle^{(q_1)} \quad \text{weakly in } L^{q_1}\big(\mathbb{R}^m\big),$$

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \langle u\rangle^{(q_2)} \quad \text{weakly in } L^{q_2}\big(\mathbb{R}^m\big). \tag{17}$$

Therefore, passing to the limit for $T \to +\infty$ in (17) yields

$$\int_{\mathbb{R}^m} \langle u\rangle^{(q_1)}\varphi(y)\,\mathrm{d}y = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \int_{\mathbb{R}^m} u\big(Y(s;y)\big)\varphi(y)\,\mathrm{d}y\,\mathrm{d}s = \int_{\mathbb{R}^m} \langle u\rangle^{(q_2)}\varphi(y)\,\mathrm{d}y$$

implying that $\langle u\rangle^{(q_1)} = \langle u\rangle^{(q_2)} \in \ker \mathcal{T}_{q_1} \cap \ker \mathcal{T}_{q_2}$. $\square$

It is possible to prove that the convergences in Proposition 2.2 are strong. This is the object of the next proposition. Actually the case $q = 2$ corresponds to the mean ergodic theorem, or von Neumann's ergodic theorem (see [15], p. 57). For the sake of completeness, proof details can be found in Appendix A.

Proposition 2.3. *Assume that $q \in (1, +\infty)$ and $u \in L^q(\mathbb{R}^m)$. Then*

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \lim_{T\to+\infty} \frac{1}{T} \int_{-T}^0 u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \langle u\rangle \quad \textit{strongly in } L^q\big(\mathbb{R}^m\big).$$

It is also possible to define the operator $\langle\cdot\rangle$ for functions in $L^1(\mathbb{R}^m)$ and $L^\infty(\mathbb{R}^m)$. These constructions are a little bit more delicate and require some additional hypotheses on the flow. As usual we introduce the relation on $\mathbb{R}^m \times \mathbb{R}^m$ given by

$$y_1 \sim y_2 \quad \text{iff } \exists s \in \mathbb{R} \text{ such that } y_2 = Y(s; y_1).$$

Using the properties of the flow it is immediate that the above relation is an equivalence relation. The classes of $\mathbb{R}^m$ with respect to $\sim$ are the orbits. For any measurable set $A \subset \mathbb{R}^m$ observe that $\mathbf{1}_A$ is constant along the flow iff A is the union of a certain subset of orbits. We will also write $\mathbf{1}_A \in \ker \mathcal{T}$ for such sets $A \subset \mathbb{R}^m$. Let us denote by $\mathcal{A}$ the family

$$\mathcal{A} = \big\{A \text{ measurable set of } \mathbb{R}^m\colon\ \mathbf{1}_A \in \ker \mathcal{T}\big\}.$$

We consider the family $\mathcal{A}_0$ of sets $A \in \mathcal{A}$ such that the only integrable function on A, constant along the flow, is the trivial one. We make the following hypothesis: there are a set $\mathcal{O} \in \mathcal{A}_0$ and a function $\xi : \mathbb{R}^m \setminus \mathcal{O} \to (0, +\infty)$ such that

$$\xi(y) = \xi\big(Y(s;y)\big), \quad s \in \mathbb{R},\ y \in \mathbb{R}^m \setminus \mathcal{O}, \qquad \int_{\mathbb{R}^m\setminus\mathcal{O}} \xi(y)\,\mathrm{d}y < +\infty. \tag{18}$$

We check easily that if such a couple $(\mathcal{O}, \xi)$ exists, then the set $\mathcal{O}$ is unique up to a negligible set. Let us analyze some examples.

Example 1. We consider $m = 2$, $b(y) = (1, 0)$. In this case we have $(Y_1, Y_2)(s; y) = (y_1 + s, y_2)$, $s \in \mathbb{R}$, $y \in \mathbb{R}^2$ and thus the constant functions along the flow are the functions depending only on y_2. We claim that $\mathcal{O} = \mathbb{R}^2$. Indeed, let $f = f(y_2) \in L^1(\mathbb{R}^2)$. Therefore we have

$$\int_{\mathbb{R}^2} \left|f(y_2)\right| \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left|f(y_2)\right| \mathrm{d}y_2 \right) \mathrm{d}y_1 < +\infty$$

implying that $\int_{\mathbb{R}} |f(y_2)|\, \mathrm{d}y_2 = 0$ which says that $f = 0$. In this case (18) is trivially satisfied.

Example 2. We consider $m = 2$, $b(y) = {}^\perp y = (y_2, -y_1)$. The flow is given by

$$Y(s; y) = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} y, \quad s \in \mathbb{R},\ y \in \mathbb{R}^2,$$

and the functions constant along the trajectories are $f = f(|y|)$. In particular $y \to \mathrm{e}^{-|y|}$ belongs to $L^1(\mathbb{R}^2)$ implying that $\mathcal{O} = \emptyset$ and that (18) holds true (with $\xi(y) = \mathrm{e}^{-|y|} > 0$ on $\mathbb{R}^2$).

Example 3. We consider $m = 2$ and $b(y) = (y_2, -\sin y_1)$. It is easily seen that $\psi(y) = \frac{1}{2}(y_2)^2 - \cos y_1$ is constant along the flow. Actually the constant functions along the trajectories are the functions depending only on $\frac{1}{2}(y_2)^2 - \cos y_1 = \psi$. We claim that $\mathcal{O} = \{y \in \mathbb{R}^2\colon\ \psi(y) > 1\} = \mathcal{O}_1 \cup \mathcal{O}_2$ where

$$\mathcal{O}_1 = \left\{y \in \mathbb{R}^2\colon\ y_2 > 2\left|\cos(y_1/2)\right|\right\}, \qquad \mathcal{O}_2 = \left\{y \in \mathbb{R}^2\colon\ y_2 < -2\left|\cos(y_1/2)\right|\right\}.$$

Indeed, let $f((y_2)^2/2 - \cos y_1)$ be a function in $L^1(\mathcal{O})$. In particular we have

$$\int_{\mathcal{O}_1} \left|f\left((y_2)^2/2 - \cos y_1\right)\right| \mathrm{d}y < +\infty.$$

Performing the change of variable $x_1 = y_1 \in \mathbb{R}$, $x_2 = (y_2)^2/2 - \cos y_1 > 1$ we obtain

$$\begin{aligned}\int_{\mathcal{O}_1} \left|f\left((y_2)^2/2 - \cos y_1\right)\right| \mathrm{d}y &= \int_{\mathbb{R}} \left(\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2 + \cos x_1)}}\, \mathrm{d}x_2 \right) \mathrm{d}x_1 \\ &\geqslant \int_{\mathbb{R}} \left(\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2 + 1)}}\, \mathrm{d}x_2 \right) \mathrm{d}x_1.\end{aligned}$$

Therefore $\int_1^{+\infty} \frac{|f(x_2)|}{\sqrt{2(x_2+1)}}\, \mathrm{d}x_2 = 0$ saying that $f((y_2)^2/2 - \cos y_1) = 0$ on $\mathcal{O}_1$. Similarly we obtain $f((y_2)^2/2 - \cos y_1) = 0$ on $\mathcal{O}_2$. Observe also that (18) holds true. Indeed we have

$$\mathbb{R}^2 \setminus \mathcal{O} = \left\{y \in \mathbb{R}^2\colon\ -1 \leqslant \psi(y) \leqslant 1\right\} = \left\{y \in \mathbb{R}^2\colon\ |y_2| \leqslant 2\left|\cos(y_1/2)\right|\right\} = \bigcup_{k \in \mathbb{Z}} A_k$$

where

$$A_k = A + (2\pi k, 0), \qquad A = \left\{ y \in [-\pi, \pi) \times \mathbb{R}:\ |y_2| \leqslant 2\left|\cos(y_1/2)\right| \right\}$$

and $\int_A \mathrm{d}y = 16$. Therefore we can consider the function

$$\xi(y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \mathbf{1}_{A_k}(y)$$

which is strictly positive on $\mathbb{R}^2 \setminus \mathcal{O}$, is constant along the flow and

$$\int_{\mathbb{R}^2 \setminus \mathcal{O}} \xi(y)\,\mathrm{d}y = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} \cdot 16 = 48 < +\infty.$$

Under the hypothesis (18) we have, for $q = 1$, a similar results as those in Proposition 2.2. The proof follows by approximating L^1 norm with L^q norms when $q \searrow 1$ (see Appendix A for details).

Proposition 2.4. *Assume that* (18) *holds and* $u \in L^1(\mathbb{R}^m)$. *Then there is a unique function* $\langle u \rangle \in \ker \mathcal{T}_1$ *such that* $\langle u \rangle|_{\mathcal{O}} = 0$ *and for any* $\varphi \in \ker \mathcal{T}_\infty$ *we have*

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} \big(u(y) - \langle u \rangle(y)\big)\varphi(y)\,\mathrm{d}y = 0. \tag{19}$$

Moreover we have the inequality $\|\langle u \rangle\|_{L^1(\mathbb{R}^m)} \leqslant \|u\|_{L^1(\mathbb{R}^m)}$. *In particular the application* $u \in L^1(\mathbb{R}^m) \to \langle u \rangle \in L^1(\mathbb{R}^m)$ *is linear, continuous and* $\|\langle \cdot \rangle\|_{\mathcal{L}(L^1(\mathbb{R}^m), L^1(\mathbb{R}^m))} \leqslant 1$.

Employing similar arguments as those in the proof of Proposition 2.2 we analyze the operator $\langle \cdot \rangle$ in the $L^\infty(\mathbb{R}^m)$ setting (see Appendix A).

Proposition 2.5. *Assume that* (18) *holds and* $u \in L^\infty(\mathbb{R}^m)$. *Then there is a unique function* $\langle u \rangle \in \ker \mathcal{T}_\infty$ *such that* $\langle u \rangle = 0$ *on* $\mathcal{O}$ *and for any* $\varphi \in \ker \mathcal{T}_1$ *we have*

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} \big(u(y) - \langle u \rangle(y)\big)\varphi(y)\,\mathrm{d}y = 0.$$

Moreover we have the weak $\star$ *convergence in* $L^\infty(\mathbb{R}^m \setminus \mathcal{O})$

$$\langle u \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s$$

and the inequality $\|\langle u \rangle\|_{L^\infty(\mathbb{R}^m)} \leqslant \|u\|_{L^\infty(\mathbb{R}^m)}$. *In particular the application* $u \in L^\infty(\mathbb{R}^m) \to \langle u \rangle \in L^\infty(\mathbb{R}^m)$ *is linear, continuous and* $\|\langle \cdot \rangle\|_{\mathcal{L}(L^\infty(\mathbb{R}^m), L^\infty(\mathbb{R}^m))} \leqslant 1$.

We inquire now about the symmetry between the operators $\langle \cdot \rangle^{(q)}$, $\langle \cdot \rangle^{(q')}$ when q, q' are conjugate exponents. We have the natural duality result.

Proposition 2.6.

a) *Assume that* $q, q' \in (1, +\infty)$, $1/q + 1/q' = 1$, $u \in L^q(\mathbb{R}^m)$, $\varphi \in L^{q'}(\mathbb{R}^m)$. *Then*

$$\int_{\mathbb{R}^m} u \langle \varphi \rangle^{(q')} \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle^{(q)} \varphi \, \mathrm{d}y.$$

b) *In particular* $\langle \cdot \rangle^{(2)}$ *is symmetric on* $L^2(\mathbb{R}^m)$ *and coincides with the orthogonal projection on* $\ker \mathcal{T}_2$. *Moreover we have the orthogonal decomposition* $L^2(\mathbb{R}^m) = \ker \mathcal{T}_2 \oplus \ker \langle \cdot \rangle^{(2)}$.

c) *Assume that* (18) *holds and that* $u \in L^1(\mathbb{R}^m)$, $\varphi \in L^\infty(\mathbb{R}^m)$. *We denote by* $\langle u \rangle^{(1)}$, $\langle \varphi \rangle^{(\infty)}$ *the functions constructed in Propositions* 2.4, 2.5 *respectively. Then*

$$\int_{\mathbb{R}^m} u \langle \varphi \rangle^{(\infty)} \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle^{(1)} \varphi \, \mathrm{d}y.$$

Proof. a) The function $\langle \varphi \rangle^{(q')}$ belongs to $\ker \mathcal{T}_{q'}$ and therefore

$$\int_{\mathbb{R}^m} \big(u - \langle u \rangle^{(q)}\big) \langle \varphi \rangle^{(q')} \, \mathrm{d}y = 0. \tag{20}$$

Similarly $\langle u \rangle^{(q)}$ belongs to $\ker \mathcal{T}_q$ and thus

$$\int_{\mathbb{R}^m} \big(\varphi - \langle \varphi \rangle^{(q')}\big) \langle u \rangle^{(q)} \, \mathrm{d}y = 0. \tag{21}$$

Combining (20), (21) yields

$$\int_{\mathbb{R}^m} u \langle \varphi \rangle^{(q')} \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle^{(q)} \langle \varphi \rangle^{(q')} \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle^{(q)} \varphi \, \mathrm{d}y.$$

b) When $q = 2$ we obtain

$$\int_{\mathbb{R}^m} u \langle \varphi \rangle^{(2)} \, \mathrm{d}y = \int_{\mathbb{R}^m} \langle u \rangle^{(2)} \varphi \, \mathrm{d}y, \quad \forall u, \varphi \in L^2\big(\mathbb{R}^m\big).$$

By the characterization in Proposition 2.2 we deduce that $\langle \cdot \rangle^{(2)} = \mathrm{Proj}_{\ker \mathcal{T}_2}$. Since $\ker \mathcal{T}_2$ is closed we have the orthogonal decomposition

$$L^2\big(\mathbb{R}^m\big) = \ker \mathcal{T}_2 \oplus (\ker \mathcal{T}_2)^\perp = \ker \mathcal{T}_2 \oplus \ker \langle \cdot \rangle^{(2)}.$$

c) By Proposition 2.4 we know that

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} \big(u - \langle u \rangle^{(1)}\big) \langle \varphi \rangle^{(\infty)} \, \mathrm{d}y = 0.$$

By construction we have $\langle \varphi \rangle^{(\infty)} = 0$ on $\mathcal{O}$ and thus we have also

$$\int_{\mathbb{R}^m} \left(u - \langle u\rangle^{(1)}\right)\langle\varphi\rangle^{(\infty)} \,\mathrm{d}y = 0.$$

By Proposition 2.5 we deduce that

$$\int_{\mathbb{R}^m\setminus\mathcal{O}} \left(\varphi - \langle\varphi\rangle^{(\infty)}\right)\langle u\rangle^{(1)} \,\mathrm{d}y = 0.$$

Since $\langle u\rangle^{(1)} = 0$ on $\mathcal{O}$, the above equality can be written

$$\int_{\mathbb{R}^m} \left(\varphi - \langle\varphi\rangle^{(\infty)}\right)\langle u\rangle^{(1)} \,\mathrm{d}y = 0.$$

Finally we obtain

$$\int_{\mathbb{R}^m} u\langle\varphi\rangle^{(\infty)} \,\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(1)}\langle\varphi\rangle^{(\infty)} \,\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle^{(1)}\varphi \,\mathrm{d}y. \qquad \square$$

The following result is a straightforward consequence of the characterizations for $\langle\cdot\rangle^{(r)}$ with $r \in [1, +\infty]$.

Corollary 2.4. *Let $u \in L^p(\mathbb{R}^m)$, $v \in L^q(\mathbb{R}^m)$ and $1/r = 1/p + 1/q$ with $p, q, r \in [1, +\infty]$. Assume that u is constant along the flow. Then $\langle uv\rangle^{(r)} = u\langle v\rangle^{(q)}$.*

Proof. We distinguish several cases.

a) $p, q, r \in (1, +\infty)$. Take any function $\varphi \in \ker \mathcal{T}_{r'}$ (with $1/r + 1/r' = 1$) and observe that $\varphi u \in \ker \mathcal{T}_{q'}$ (with $1/q + 1/q' = 1$). Therefore we know that

$$\int_{\mathbb{R}^m} \left(v - \langle v\rangle^{(q)}\right)\varphi u \,\mathrm{d}y = 0$$

saying that $\langle uv\rangle^{(r)} = u\langle v\rangle^{(q)}$.

b) $r \in (1, +\infty)$, $p = r$, $q = +\infty$ (we assume that (18) holds). For any function $\varphi \in \ker \mathcal{T}_{r'}$ we have $\varphi u \in \ker \mathcal{T}_1$ and thus

$$\int_{\mathbb{R}^m\setminus\mathcal{O}} \left(v - \langle v\rangle^{(\infty)}\right)\varphi u \,\mathrm{d}y = 0.$$

Since $\varphi u = 0$ on $\mathcal{O}$ (as function in $\ker \mathcal{T}_1$) we deduce that

$$\int_{\mathbb{R}^m} \left(v - \langle v\rangle^{(\infty)}\right)\varphi u \,\mathrm{d}y = 0$$

implying that $\langle uv\rangle^{(r)} = u\langle v\rangle^{(\infty)}$.

The other cases are: c) $r \in (1, +\infty)$, $p = +\infty$, $q = r$, d) $r = 1$, $p, q \in (1, +\infty)$, e) $r = p = 1$, $q = +\infty$, f) $r = q = 1$, $p = +\infty$, g) $r = p = q = +\infty$. They follow in similar way and are left to the reader. $\square$

Recall that our main motivation when constructing the average operator was that the equality $\langle b \cdot \nabla_y u_1 \rangle = 0$ holds true for any function u_1, making thus possible to eliminate the first order correction term in (4) after applying the average operator. In the sequel we justify rigorously this property. By the orthogonal decomposition in Proposition 2.6 we deduce that $\ker\langle\cdot\rangle^{(2)} = (\ker \mathcal{T}_2)^{\perp} = (\ker \mathcal{T}_2^{\star})^{\perp} = \overline{\operatorname{range} \mathcal{T}_2}$. We have the general result.

Proposition 2.7. *Assume that $q \in (1, +\infty)$. Then $\ker\langle\cdot\rangle^{(q)} = \overline{\operatorname{range} \mathcal{T}_q}$.*

Proof. For any $v = \mathcal{T}_q u \in \operatorname{range} \mathcal{T}_q$ and $\varphi \in \ker \mathcal{T}_{q'}$ we have

$$\int_{\mathbb{R}^m} (v - 0)\varphi \,\mathrm{d}y = \int_{\mathbb{R}^m} \mathcal{T}_q u \varphi \,\mathrm{d}y = -\int_{\mathbb{R}^m} u \mathcal{T}_{q'} \varphi \,\mathrm{d}y = 0$$

saying that $\langle v \rangle^{(q)} = 0$. Therefore $\operatorname{range} \mathcal{T}_q \subset \ker\langle\cdot\rangle^{(q)}$ and also $\overline{\operatorname{range} \mathcal{T}_q} \subset \ker\langle\cdot\rangle^{(q)}$. Consider now a linear form h on $L^q(\mathbb{R}^m)$ vanishing on $\operatorname{range} \mathcal{T}_q$. There is $v \in L^{q'}(\mathbb{R}^m)$ such that $h(w) = \int_{\mathbb{R}^m} wv \,\mathrm{d}y$ for any $w \in L^q(\mathbb{R}^m)$. In particular

$$\int_{\mathbb{R}^m} \mathcal{T}_q u v \,\mathrm{d}y = 0, \quad \forall u \in \mathrm{D}_q,$$

saying that $v \in \ker \mathcal{T}_{q'}$. For any $\varphi \in \ker\langle\cdot\rangle^{(q)}$ we can write by Proposition 2.6

$$h(\varphi) = \int_{\mathbb{R}^m} v\varphi \,\mathrm{d}y = \int_{\mathbb{R}^m} \langle v \rangle^{(q')} \varphi \,\mathrm{d}y = \int_{\mathbb{R}^m} v \langle \varphi \rangle^{(q)} \,\mathrm{d}y = 0$$

and thus h vanishes on $\ker\langle\cdot\rangle^{(q)}$ implying that $\ker\langle\cdot\rangle^{(q)} \subset \overline{\operatorname{range} \mathcal{T}_q}$. Consequently we have $\overline{\operatorname{range} \mathcal{T}_q} = \ker\langle\cdot\rangle^{(q)}$. □

At this stage let us point out that if $\operatorname{range} \mathcal{T}_q$ is closed, then $\ker\langle\cdot\rangle^{(q)} = \operatorname{range} \mathcal{T}_q$ saying that the equation $\mathcal{T}_q u = f \in L^q(\mathbb{R}^m)$ is solvable iff $\langle f \rangle^{(q)} = 0$. Let us indicate a simple situation where the above characterization for the range of $\mathcal{T}_q$ occurs.

Proposition 2.8. *Assume that all the trajectories are closed, uniformly in time, i.e.,*

$$\exists T > 0: \quad \forall y \in \mathbb{R}^m, \ \exists T_y \in [0, T] \text{ such that } Y(T_y; y) = y.$$

Then for any $q \in (1, +\infty)$ the range of $\mathcal{T}_q$ is closed and we have $\operatorname{range} \mathcal{T}_q = \ker\langle\cdot\rangle^{(q)}$.

Proof. By Proposition 2.7 we have $\operatorname{range} \mathcal{T}_q \subset \overline{\operatorname{range} \mathcal{T}_q} = \ker\langle\cdot\rangle^{(q)}$. Conversely, assume that $f \in \ker\langle\cdot\rangle^{(q)}$ and let us check that $f \in \operatorname{range} \mathcal{T}_q$. For any $\mu > 0$ let $u_\mu \in L^q(\mathbb{R}^m)$ solving

$$\mu u_\mu + \mathcal{T}_q u_\mu = f. \tag{22}$$

It is easily seen that the unique solution of the above equation is

$$u_\mu = \int_{-\infty}^{0} \mathrm{e}^{\mu s} f\big(Y(s; \cdot)\big) \,\mathrm{d}s. \tag{23}$$

Observe that we are done if we prove that $(\|u_\mu\|_{L^q(\mathbb{R}^m)})_{\mu>0}$ is bounded. Indeed, in this case we can extract a sequence $(\mu_n)_n$ converging towards 0 such that $\lim_{n\to+\infty} u_{\mu_n} = u$ weakly in $L^q(\mathbb{R}^m)$. Passing to the limit in the weak formulation of (22) we deduce that $u \in \mathrm{D}_q$ and $f = \mathcal{T}_q u \in \operatorname{range} \mathcal{T}_q$. In order to estimate $(\|u_\mu\|_{L^q(\mathbb{R}^m)})_{\mu>0}$ we use the immediate lemma, whose proof is left to the reader.

Lemma 2.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a locally integrable T periodic function. Then for any $t \in \mathbb{R}^\star$ we have*

$$\left| \frac{1}{t} \int_0^t g(s)\,\mathrm{d}s - \frac{1}{T} \int_0^T g(s)\,\mathrm{d}s \right| \leqslant \frac{2}{|t|} \int_0^T \left|g(s)\right| \mathrm{d}s.$$

By Proposition 2.3 we know that

$$\lim_{s\to-\infty} \left(-\frac{1}{s} \int_s^0 f\big(Y(\tau;\cdot)\big)\,\mathrm{d}\tau \right) = \langle f \rangle^{(q)} = 0, \quad \text{strongly in } L^q\big(\mathbb{R}^m\big).$$

In particular we have the pointwise convergence

$$\lim_{k\to+\infty} \left(-\frac{1}{s_k} \int_{s_k}^0 f\big(Y(\tau;y)\big)\,\mathrm{d}\tau \right) = 0, \quad \text{a.e. } y \in \mathbb{R}^m,$$

for some sequence $(s_k)_k$ verifying $\lim_{k\to+\infty} s_k = -\infty$. Observe that

$$\left\| \int_0^T \left| f\big(Y(\tau;\cdot)\big)\right| \mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \leqslant T \|f\|_{L^q(\mathbb{R}^m)} < +\infty$$

and thus, for a.a. $y \in \mathbb{R}^m$ the function $\tau \to f(Y(\tau;y))$ is locally integrable. Since the function $\tau \to f(Y(\tau;y))$ is T_y periodic, we have by Lemma 2.1

$$\frac{1}{T_y} \int_0^{T_y} f\big(Y(\tau;y)\big)\,\mathrm{d}\tau = \lim_{k\to+\infty} \left(-\frac{1}{s_k} \int_{s_k}^0 f\big(Y(\tau;y)\big)\,\mathrm{d}\tau \right) = 0, \quad \text{a.e. } y \in \mathbb{R}^m,$$

and

$$\begin{aligned}
\left\| -\frac{1}{s} \int_s^0 f\big(Y(\tau;\cdot)\big)\,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} &\leqslant \left\| \frac{2}{|s|} \int_0^{T_y} \left| f\big(Y(\tau;y)\big)\right| \mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \\
&\leqslant \left\| \frac{2}{|s|} \int_0^{T} \left| f\big(Y(\tau;y)\big)\right| \mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \\
&\leqslant \frac{2T}{|s|} \|f\|_{L^q(\mathbb{R}^m)}
\end{aligned}$$

implying that

$$\left\| \int_s^0 f\big(Y(\tau;\cdot)\big)\,\mathrm{d}\tau \right\|_{L^q(\mathbb{R}^m)} \leqslant 2T\|f\|_{L^q(\mathbb{R}^m)}. \tag{24}$$

Coming back to (23) one gets after integration by parts

$$u_\mu = -\int_{-\infty}^0 \mathrm{e}^{\mu s}\frac{d}{ds}\left\{\int_s^0 f\big(Y(\tau;\cdot)\big)\,\mathrm{d}\tau\right\}\mathrm{d}s = \int_{-\infty}^0 \mu\mathrm{e}^{\mu s}\int_s^0 f\big(Y(\tau;\cdot)\big)\,\mathrm{d}\tau\,\mathrm{d}s$$

and therefore, combining with (24) yields

$$\|u_\mu\|_{L^q(\mathbb{R}^m)} \leqslant 2T\|f\|_{L^q(\mathbb{R}^m)}, \quad \forall\mu > 0. \qquad \square$$

Remark 2.2. The hypothesis that all trajectories are closed, uniformly in time, implies a Poincaré inequality. Indeed, taking the weak limit of the previous uniform inequality yields $\|u\|_{L^q(\mathbb{R}^m)} \leqslant 2T\|f\|_{L^q(\mathbb{R}^m)}$. Taking the average of (22) it is easily seen that $\langle u_\mu\rangle^{(q)} = 0$, $\mu > 0$ and therefore $\langle u\rangle^{(q)} = 0$. Finally we obtain

$$\|u\|_{L^q(\mathbb{R}^m)} \leqslant 2T\|\mathcal{T}_q u\|_{L^q(\mathbb{R}^m)}, \quad \langle u\rangle = 0.$$

Remark 2.3. The hypotheses in Proposition 2.8 are verified in the case of a periodic flow, with uniformly bounded periods. This happens to be true for the guiding-center approximation (see Lemma 5.1) and for the finite Larmor radius regime with constant magnetic field.

Generally the flow is not uniformly periodic. For later use we establish here a characterization for $\ker\langle\cdot\rangle^{(q)} = \overline{\operatorname{range}\mathcal{T}_q}$ in the general case (not necessarily periodic). This result will be useful when justifying the asymptotic behavior of (1) when $\varepsilon \searrow 0$.

Proposition 2.9. *Let f be a function in $L^q(\mathbb{R}^m)$ for some $q \in (1, +\infty)$. For any $\mu > 0$ we denote by u_μ the unique solution of* (22)*. Then the following statements are equivalent*:

a) $\langle f\rangle^{(q)} = 0$.
b) $\lim_{\mu\searrow 0}(\mu u_\mu) = 0$ *in* $L^q(\mathbb{R}^m)$.

Proof. Assume that b) holds true. Applying the operator $\langle\cdot\rangle^{(q)}$ in (22) one gets

$$\langle f\rangle^{(q)} = \langle \mu u_\mu\rangle^{(q)} + \langle \mathcal{T}_q u_\mu\rangle^{(q)} = \langle \mu u_\mu\rangle^{(q)}, \quad \forall\mu > 0,$$

and therefore

$$\langle f\rangle^{(q)} = \lim_{\mu\searrow 0}\langle \mu u_\mu\rangle^{(q)} = \Big\langle \lim_{\mu\searrow 0}(\mu u_\mu)\Big\rangle^{(q)} = 0.$$

Conversely, suppose that a) holds true. Considering the function $G(s;y) = \int_s^0 f(Y(\tau;y))\,\mathrm{d}\tau$ we obtain by the formula (23) (use the inequality $\|G(s;\cdot)\|_{L^q(\mathbb{R}^m)} \leqslant |s|\|f\|_{L^q(\mathbb{R}^m)}$ in order to justify the integration by parts)

$$u_\mu = -\int_{-\infty}^0 \mathrm{e}^{\mu s}\frac{\partial G}{\partial s}(s;\cdot)\,\mathrm{d}s = \int_{-\infty}^0 \mu s\mathrm{e}^{\mu s}\frac{G(s;\cdot)}{s}\,\mathrm{d}s = \frac{1}{\mu}\int_{-\infty}^0 t\mathrm{e}^t\frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}}\,\mathrm{d}t.$$

We know that $\|G(t\mu^{-1})/(t\mu^{-1})\|_{L^q(\mathbb{R}^m)} \leqslant \|f\|_{L^q(\mathbb{R}^m)}$ and by Proposition 2.3 we have for any $t<0$

$$\lim_{\mu\searrow 0}\frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}} = \lim_{\mu\searrow 0}\frac{\int_{t/\mu}^{0} f(Y(s;\cdot))\,\mathrm{d}s}{t/\mu} = -\langle f\rangle^{(q)} = 0, \quad \text{strongly in } L^q\big(\mathbb{R}^m\big).$$

Consequently, by the dominated convergence theorem, one gets

$$\|\mu u_\mu\|_{L^q(\mathbb{R}^m)} \leqslant \int_{-\infty}^{0} |t|\mathrm{e}^t \left\| \frac{G(t\mu^{-1};\cdot)}{t\mu^{-1}} \right\|_{L^q(\mathbb{R}^m)} \mathrm{d}t \to 0 \quad \text{as } \mu\searrow 0. \qquad \square$$

Remark 2.4. With the above notations we have $\|\mu u_\mu\|_{L^q(\mathbb{R}^m)} \leqslant \|f\|_{L^q(\mathbb{R}^m)}$, $\forall\mu>0$.

2.1. Average operator commutator with time/space derivatives

Up to this point we have investigated the properties of $\langle\cdot\rangle^{(q)}$ operating from $L^q(\mathbb{R}^m)$ to $L^q(\mathbb{R}^m)$ with $q\in[1,+\infty]$. In view of further regularity results for the transport equations (1) we investigate now how $\langle\cdot\rangle^{(q)}$ acts on some particular subspaces of smooth functions. These regularity results will be crucial when justifying the strong convergence of the solutions of (1) when $\varepsilon\searrow 0$. For this purpose we recall here the following basic results concerning the derivation operators along fields in $\mathbb{R}^m$. For any $\xi=(\xi_1(y),\dots,\xi_m(y))$, where $y\in\mathbb{R}^m$, we denote by L_ξ the operator $\xi\cdot\nabla_y$. A direct computation shows that for any smooth fields ξ,η, the commutator between L_ξ, L_η is still a first order operator, given by $[L_\xi, L_\eta] := L_\xi L_\eta - L_\eta L_\xi = L_\chi$, where χ is the Poisson bracket of ξ and η

$$\chi=[\xi,\eta], \qquad [\xi,\eta]_i = (\xi\cdot\nabla_y)\eta_i - (\eta\cdot\nabla_y)\xi_i = L_\xi(\eta_i) - L_\eta(\xi_i), \quad i\in\{1,\dots,m\}.$$

It is well known (see [3, p. 93]) that L_ξ, L_η commute (or equivalently the Poisson bracket $[\xi,\eta]$ vanishes) iff the flows corresponding to ξ,η, let us say Z_1, Z_2, commute

$$Z_1\big(s_1; Z_2(s_2;y)\big) = Z_2\big(s_2; Z_1(s_1;y)\big), \quad s_1,s_2\in\mathbb{R},\ y\in\mathbb{R}^m.$$

Consider a smooth field c in involution with b and having bounded divergence

$$c\in W^{1,\infty}_{\mathrm{loc}}\big(\mathbb{R}^m\big), \qquad \mathrm{div}_y\, c\in L^\infty\big(\mathbb{R}^m\big), \qquad [c,b]=0$$

and let us denote by Z the flow associated to c (we assume that Z is well defined for any $(s,y)\in\mathbb{R}\times\mathbb{R}^m$). We claim that the following commutation property holds true.

Proposition 2.10. *Assume that c is a smooth field in involution with b, with bounded divergence and well defined flow. Then for any $q\in(1,+\infty)$ the operator $\langle\cdot\rangle^{(q)}$ commutes with the translations along the flow of c*

$$\big\langle u\circ Z(h;\cdot)\big\rangle^{(q)} = \langle u\rangle^{(q)}\circ Z(h;\cdot), \quad u\in L^q\big(\mathbb{R}^m\big),\ h\in\mathbb{R}.$$

Moreover, under the hypothesis (18) *the above conclusion holds true when $q\in\{1,+\infty\}$.*

Proof. Assume that $q\in(1,+\infty)$. The commutation property of the flows Y, Z and Proposition 2.3 allow us to write the strong convergences in $L^q(\mathbb{R}^m)$

$$\begin{aligned}
\big\langle u \circ Z(h;\cdot)\big\rangle^{(q)} &= \lim_{T\to+\infty} \frac{1}{T}\int_0^T u \circ Z\big(h; Y(s;\cdot)\big)\,\mathrm{d}s \\
&= \lim_{T\to+\infty} \frac{1}{T}\int_0^T u \circ Y\big(s; Z(h;\cdot)\big)\,\mathrm{d}s \\
&= \left(\lim_{T\to+\infty} \frac{1}{T}\int_0^T u\big(Y(s;\cdot)\big)\,\mathrm{d}s\right) \circ Z(h;\cdot) \\
&= \langle u\rangle^{(q)} \circ Z(h;\cdot).
\end{aligned} \tag{25}$$

Notice that the third equality in the above computations follows by changing the variable along the flow Z and by using the boundedness of $\operatorname{div}_y c$. The cases $q \in \{1, +\infty\}$ require more careful analysis. The idea is to appeal to the variational characterization of the average operator in Propositions 2.4, 2.5 by choosing appropriate test functions invariant along the flow of b and then performing changes of variable along the flow of c. The main point here is that the divergence of any smooth field in involution with b (in particular $\operatorname{div}_y c$) is invariant along the flow of b. The details are left to the reader. $\square$

Remark 2.5. In particular we have $[b, b] = 0$ and therefore $\langle\cdot\rangle^{(q)}$ commutes with the translations along the flow of b. We have for any $h \in \mathbb{R}$, $u \in L^q(\mathbb{R}^m)$, $q \in [1, +\infty]$

$$\big\langle u\big(Y(h;\cdot)\big)\big\rangle^{(q)} = \langle u\rangle^{(q)}\big(Y(h;\cdot)\big) = \langle u\rangle^{(q)}.$$

We will show that for any smooth field c in involution with b, the operator $\langle\cdot\rangle^{(q)}$ commutes with $c \cdot \nabla_y$. We denote by $\mathcal{T}_q^c$ the operator given by

$$\mathrm{D}\big(\mathcal{T}_q^c\big) = \big\{u \in L^q\big(\mathbb{R}^m\big):\ \operatorname{div}_y(cu) \in L^q\big(\mathbb{R}^m\big)\big\}, \qquad \mathcal{T}_q^c u = \operatorname{div}_y(cu) - (\operatorname{div}_y c)u, \quad u \in \mathrm{D}\big(\mathcal{T}_q^c\big).$$

We have the standard result (see [6, Proposition IX.3, p. 153], for similar results).

Lemma 2.2. *Assume that $q \in (1, +\infty)$ and let u be a function in $L^q(\mathbb{R}^m)$. Then the following statements are equivalent*:

a) $u \in \mathrm{D}(\mathcal{T}_q^c)$.
b) $(h^{-1}(u(Z(h;\cdot)) - u))_h$ *is bounded in* $L^q(\mathbb{R}^m)$.

Moreover, for any $u \in \mathrm{D}(\mathcal{T}_q^c)$ we have the convergence

$$\lim_{h\to 0} \frac{u(Z(h;\cdot)) - u}{h} = \mathcal{T}_q^c u, \quad \textit{strongly in } L^q\big(\mathbb{R}^m\big).$$

The next result is a straightforward consequence of Proposition 2.10 and Lemma 2.2.

Proposition 2.11. *Under the hypotheses of Proposition* 2.10, *assume that $u \in \mathrm{D}(\mathcal{T}_q^c)$ for some $q \in (1, +\infty)$. Then $\langle u\rangle^{(q)} \in \mathrm{D}(\mathcal{T}_q^c)$ and $\mathcal{T}_q^c\langle u\rangle^{(q)} = \langle \mathcal{T}_q^c u\rangle^{(q)}$.*

Remark 2.6. In particular Proposition 2.11 applies for $c = b$. Actually, for any $u \in \mathrm{D}(\mathcal{T}_q)$, $q \in [1, +\infty]$ we have $\mathcal{T}_q\langle u\rangle^{(q)} = \langle \mathcal{T}_q u\rangle^{(q)} = 0$.

Remark 2.7. Under the hypotheses of Proposition 2.10 we check immediately thanks to Lemma 2.2 that if $u \in \mathrm{D}(\mathcal{T}_q^c)$, then for any $s \in \mathbb{R}$, $u \circ Y(s;\cdot) \in \mathrm{D}(\mathcal{T}_q^c)$ and

$$\mathcal{T}_q^c\big(u \circ Y(s;\cdot)\big) = \big(\mathcal{T}_q^c u\big) \circ Y(s;\cdot).$$

In particular if $u \in \ker \mathcal{T}_q \cap \mathrm{D}(\mathcal{T}_q^c)$ then $\mathcal{T}_q^c u \in \ker \mathcal{T}_q$.

The last result in this section states that $\langle \cdot \rangle^{(q)}$ commutes with the time derivation. The proof is standard and comes easily by observing that

$$\frac{\langle u(t+h)\rangle^{(q)} - \langle u(t)\rangle^{(q)}}{h} = \left\langle \frac{u(t+h)-u(t)}{h} \right\rangle^{(q)}$$

and by adapting the arguments in Lemma 2.2.

Proposition 2.12. *Assume that $u \in W^{1,p}([0,T]; L^q(\mathbb{R}^m))$ for some $p, q \in (1,+\infty)$. Then the application $(t,y) \to \langle u(t,\cdot)\rangle^{(q)}(y)$ belongs to $W^{1,p}([0,T]; L^q(\mathbb{R}^m))$ and we have $\partial_t \langle u\rangle^{(q)} = \langle \partial_t u\rangle^{(q)}$.*

3. Well-posedness of the limit model

We continue our mathematical analysis by studying the well-posedness of the limit model and surely, one of the key point will be to justify rigorously the asymptotic behavior towards this limit model. These items will be carried out in the next sections. This section is devoted to the study of the limit model, when $\varepsilon \searrow 0$, for the transport problems (1). Recall that b is a given smooth field satisfying (10)–(12). We assume that a satisfies the conditions

$$a \in L^1\big([0,T]; W^{1,\infty}\big(\mathbb{R}^m\big)\big), \qquad \mathrm{div}_y\, a = 0. \tag{26}$$

Based on Hilbert's expansion method we have obtained (see (3), (4)) the formula $u^\varepsilon = u + \varepsilon u_1 + \mathcal{O}(\varepsilon^2)$ where

$$b(y)\cdot\nabla_y u = 0, \qquad \partial_t u + a(t,y)\cdot\nabla_y u + b(y)\cdot\nabla_y u_1 = 0.$$

Projecting the second equation on the kernel of $\mathcal{T}$ leads to the model

$$\partial_t\langle u\rangle + \big\langle a(t)\cdot\nabla_y u(t)\big\rangle = 0, \quad (t,y)\in(0,T)\times\mathbb{R}^m.$$

Notice that $\mathcal{T}u = 0$ and thus $\langle u\rangle = u$. Finally we obtain

$$\begin{cases} \partial_t u + \big\langle a(t)\cdot\nabla_y u(t)\big\rangle = 0, \quad b(y)\cdot\nabla_y u = 0, & (t,y)\in(0,T)\times\mathbb{R}^m, \\ u(0,y) = u_0(y), & y \in \mathbb{R}^m. \end{cases} \tag{27}$$

We work in the $L^q(\mathbb{R}^m)$ setting, with $q \in (1,+\infty)$. For any $\varphi \in \ker \mathcal{T}_{q'}$ we have

$$\int_{\mathbb{R}^m} \Big(a(t,y)\cdot\nabla_y u - \big\langle a(t)\cdot\nabla_y u(t)\big\rangle^{(q)}\Big)\varphi(y)\,\mathrm{d}y = 0$$

and we introduce the notion of weak solution for (27) as follows.

Definition 3.1. Assume that $u_0 \in \ker \mathcal{T}_q$, $f \in L^1([0,T]; \ker \mathcal{T}_q)$ (*i.e.*, $f \in L^1([0,T]; L^q(\mathbb{R}^m))$ and $f(t) \in \ker \mathcal{T}_q$, $t \in [0,T]$). We say that $u \in L^\infty([0,T]; \ker \mathcal{T}_q)$ is a weak solution for

$$\begin{cases} \partial_t u + \langle a(t) \cdot \nabla_y u(t) \rangle^{(q)} = f(t,y), \quad \mathcal{T}_q u = 0, & (t,y) \in (0,T) \times \mathbb{R}^m, \\ u(0,y) = u_0(y), & y \in \mathbb{R}^m, \end{cases} \tag{28}$$

iff for any $\varphi \in C_c^1([0,T) \times \mathbb{R}^m)$ satisfying $\mathcal{T}\varphi = 0$ we have

$$\int_0^T \int_{\mathbb{R}^m} u(t,y)\big(\partial_t \varphi + \mathrm{div}_y(\varphi a)\big)\,\mathrm{d}y\,\mathrm{d}t + \int_{\mathbb{R}^m} u_0(y)\varphi(0,y)\,\mathrm{d}y + \int_0^T \int_{\mathbb{R}^m} f(t,y)\varphi(t,y)\,\mathrm{d}y\,\mathrm{d}t = 0. \tag{29}$$

We start by establishing existence and regularity results for the solution of (28).

Proposition 3.1. *Assume that $u_0 \in \ker \mathcal{T}_q$, $f \in L^1([0,T]; \ker \mathcal{T}_q)$ for some $q \in (1, +\infty)$. Then there is at least a weak solution $u \in L^\infty([0,T]; \ker \mathcal{T}_q)$ of (28) satisfying*

$$\big\|u(t)\big\|_{L^q(\mathbb{R}^m)} \leqslant \|u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \big\|f(s)\big\|_{L^q(\mathbb{R}^m)}, \quad t \in [0,T].$$

Moreover, if $u_0 \geqslant 0$ and $f \geqslant 0$ then $u \geqslant 0$.

Proof. For any $\varepsilon > 0$ there is a unique weak solution u^ε of

$$\begin{cases} \partial_t u^\varepsilon + a(t,y) \cdot \nabla_y u^\varepsilon + \dfrac{b(y)}{\varepsilon} \cdot \nabla_y u^\varepsilon = f(t,y), & (t,y) \in (0,T) \times \mathbb{R}^m, \\ u^\varepsilon(0,y) = u_0(y), & y \in \mathbb{R}^m. \end{cases} \tag{30}$$

The solution is given by

$$u^\varepsilon(t,y) = u_0\big(Z^\varepsilon(0;t,y)\big) + \int_0^t f\big(s, Z^\varepsilon(s;t,y)\big)\,\mathrm{d}s, \quad (t,y) \in [0,T] \times \mathbb{R}^m,$$

where Z^ε are the characteristics corresponding to the field $a + \varepsilon^{-1} b$. Multiplying by $u^\varepsilon(t,y)|u^\varepsilon(t,y)|^{q-2}$ and integrating with respect to $y \in \mathbb{R}^m$, we obtain thanks to Hölder's inequality

$$\big\|u^\varepsilon\big\|_{L^q(\mathbb{R}^m)} \leqslant \|u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \big\|f(s)\big\|_{L^q(\mathbb{R}^m)}, \quad t \in [0,T].$$

We extract a sequence $(\varepsilon_k)_k$ converging towards 0 such that $u^{\varepsilon_k} \rightharpoonup u$ weakly $\star$ in $L^\infty([0,T]; L^q(\mathbb{R}^m))$ for some function $u \in L^\infty([0,T]; L^q(\mathbb{R}^m))$ satisfying

$$\|u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leqslant \|u_0\|_{L^q(\mathbb{R}^m)} + \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))}.$$

By the weak formulation of (30) with a function $\varphi \in C_c^1([0,T) \times \mathbb{R}^m)$ we deduce that

$$\int_0^T \int_{\mathbb{R}^m} u^{\varepsilon_k} \left(\partial_t \varphi + \left(a + \frac{b}{\varepsilon_k} \right) \cdot \nabla_y \varphi \right) \mathrm{d}y \, \mathrm{d}t + \int_{\mathbb{R}^m} u_0 \varphi(0, y) \, \mathrm{d}y + \int_0^T \int_{\mathbb{R}^m} f \varphi \, \mathrm{d}y \, \mathrm{d}t = 0. \tag{31}$$

Multiplying by ε_k and passing to the limit as $k \to +\infty$ one gets easily by Proposition 2.1 that $u(t) \in \ker \mathcal{T}_q$, $t \in [0, T)$. If the test function verifies $\mathcal{T}\varphi = 0$ we get rid of the singular term in (31) and by passing to the limit for $k \to +\infty$ we deduce that the weak $\star$ limit u satisfies the weak formulation of (28). If $u_0 \geqslant 0$, $f \geqslant 0$ then $u^\varepsilon \geqslant 0$ for any $\varepsilon > 0$ and thus the solution constructed above is non-negative. $\square$

At this stage we mention that the numerical approximation of the limit model (28) remains a difficult problem. The main drawback of the weak formulation (29) is the particular form of the trial functions $\varphi \in \ker \mathcal{T} \cap C_c^1([0, T) \times \mathbb{R}^m)$. Generally, the choice of such test functions could be a difficult task. Accordingly, we are looking for a strong formulation of (28). Therefore we inquire about the smoothness of the solution. We also mention that the regularity results will allow us to prove strong convergence results for the solutions of (1) towards the solution of (28) as $\varepsilon \searrow 0$. A complete regularity analysis can be carried out under the following hypothesis: we will assume that the field a is a linear combination of fields in involution with $b^0 := b$

$$a(t, y) = \sum_{i=0}^{r} \alpha_i(t, y) b^i(y), \quad b^i \in W^{1,\infty}\big(\mathbb{R}^m\big), \ \big[b^i, b\big] = 0, \ i \in \{1, \dots, r\}, \tag{32}$$

where $(\alpha_i)_i$ are smooth coefficients verifying

$$\alpha_i \in L^1\big([0, T]; L^\infty\big(\mathbb{R}^m\big)\big), \qquad b^j \cdot \nabla_y \alpha_i \in L^1\big([0, T]; L^\infty\big(\mathbb{R}^m\big)\big), \quad i, j \in \{0, 1, \dots, r\}. \tag{33}$$

For any $i \in \{1, \dots, r\}$ we denote by $\mathcal{T}_q^i : \mathrm{D}(\mathcal{T}_q^i) \subset L^q(\mathbb{R}^m) \to L^q(\mathbb{R}^m)$ the operator

$$\mathrm{D}\big(\mathcal{T}_q^i\big) = \big\{u \in L^q\big(\mathbb{R}^m\big) \colon \operatorname{div}_y\big(b^i u\big) \in L^q\big(\mathbb{R}^m\big)\big\}, \qquad \mathcal{T}_q^i u = \operatorname{div}_y\big(b^i u\big) - \big(\operatorname{div}_y b^i\big) u, \quad u \in \mathrm{D}\big(\mathcal{T}_q^i\big),$$

and by Y^i the flow associated to b^i. Since $[b^i, b] = 0$ then Y^i commutes with Y for any $i \in \{1, \dots, r\}$. Under the previous hypotheses it can be shown that the weak solution constructed in Proposition 3.1 propagates the regularity of the initial condition. The proof is rather technical and it is postponed to Appendix B.

Proposition 3.2. *Assume that* (32), (33) *hold,* $u_0 \in \ker \mathcal{T}_q \cap (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i))$, $f \in L^1([0, T]; \ker \mathcal{T}_q \cap (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)))$ *(i.e.,* $f \in L^1([0, T]; L^q(\mathbb{R}^m))$, $\mathcal{T}_q f = 0$ *and* $\mathcal{T}_q^i f \in L^1([0, T]; L^q(\mathbb{R}^m))$, $i \in \{1, \dots, r\}$*) and let us denote by* u *the weak solution of* (28) *constructed in Proposition* 3.1. *Then we have* $u(t) \in \ker \mathcal{T}_q \cap (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i))$, $t \in [0, T]$, *and*

$$\begin{aligned} &\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r \big\|\mathcal{T}_q^i u\big\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \\ &\quad \leqslant C\left(\|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r \big\|\mathcal{T}_q^i f\big\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r \big\|\mathcal{T}_q^i u_0\big\|_{L^q(\mathbb{R}^m)}\right) \end{aligned}$$

for some constant depending on $\sum_{0 \leqslant i, j \leqslant r} \|b^i \cdot \nabla_y \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{i=0}^r \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$. *Moreover, if* $f \in L^\infty([0, T]; L^q(\mathbb{R}^m))$, $\alpha_i \in L^\infty([0, T]; L^\infty(\mathbb{R}^m))$ *for any* $i \in \{1, \dots, r\}$ *then* $\partial_t u \in L^\infty([0, T]; L^q(\mathbb{R}^m))$.

Thanks to the previous regularity result we are able to establish the existence of strong solution for (28).

Definition 3.2. Under the hypotheses (32), (33), (18) we say that u is a strong solution of (28) iff $u \in L^\infty([0,T]; L^q(\mathbb{R}^m))$, $\partial_t u \in L^1([0,T]; L^q(\mathbb{R}^m))$, $\mathcal{T}_q^i u \in L^\infty([0,T]; L^q(\mathbb{R}^m))$ for any $i \in \{1,\dots,r\}$ and

$$\begin{cases} \partial_t u + \sum_{i=1}^{r} \langle \alpha_i(t) \rangle^{(\infty)} \mathcal{T}_q^i u(t) = f(t), \quad \mathcal{T}_q u(t) = 0, \quad t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{34}$$

Corollary 3.1. *Assume that* (32), (33), (18) *hold. Then for any* $u_0 \in (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)) \cap \ker \mathcal{T}_q$ *and* $f \in L^1([0,T]; (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)) \cap \ker \mathcal{T}_q)$, *there is a strong solution u for* (28) *verifying*

$$\begin{aligned} &\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^{r} \left\| \mathcal{T}_q^i u \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \\ &\quad \leqslant C\|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + C \sum_{i=1}^{r} \left\{ \left\| \mathcal{T}_q^i f \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \left\| \mathcal{T}_q^i u_0 \right\|_{L^q(\mathbb{R}^m)} \right\}. \end{aligned} \tag{35}$$

Proof. Let u be the solution constructed in Proposition 3.2. This function has the regularity in (35), satisfies $\mathcal{T}_q u = 0$ and

$$\int_0^T \int_{\mathbb{R}^m} u\big(\partial_t \varphi + \mathrm{div}_y(\varphi a)\big)\,\mathrm{d}y\,\mathrm{d}t + \int_{\mathbb{R}^m} u_0 \varphi(0,y)\,\mathrm{d}y + \int_0^T \int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{36}$$

for any function $\varphi \in C_c^1([0,T) \times \mathbb{R}^m)$ verifying $\mathcal{T}\varphi = 0$. Since $a = \sum_{i=0}^r \alpha_i b^i$ and $\mathcal{T}_q u = 0$ one gets

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^m} u\,\mathrm{div}_y(a\varphi)\,\mathrm{d}y\,\mathrm{d}t &= \int_0^T \int_{\mathbb{R}^m} u\,\mathrm{div}_y\left(\varphi \sum_{i=0}^{r} \alpha_i b^i\right)\mathrm{d}y\,\mathrm{d}t \\ &= -\sum_{i=1}^{r} \int_0^T \int_{\mathbb{R}^m} \alpha_i \varphi \mathcal{T}_q^i u\,\mathrm{d}y\,\mathrm{d}t \end{aligned}$$

implying that

$$\int_0^T \int_{\mathbb{R}^m} \left(\partial_t u + \sum_{i=1}^{r} \alpha_i \mathcal{T}_q^i u\right)\varphi\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^m} f\varphi\,\mathrm{d}y\,\mathrm{d}t. \tag{37}$$

Using now the properties of the operators $\langle\cdot\rangle^{(q)}$, $\langle\cdot\rangle^{(q')}$ we obtain for any $t \in [0,T]$

$$\int_{\mathbb{R}^m} \varphi(t) \sum_{i=1}^{r} \alpha_i(t) \mathcal{T}_q^i u(t)\,\mathrm{d}y = \int_{\mathbb{R}^m} \big\langle \varphi(t) \big\rangle^{(q')} \sum_{i=1}^{r} \alpha_i(t) \mathcal{T}_q^i u(t)\,\mathrm{d}y$$
$$= \sum_{i=1}^{r} \int_{\mathbb{R}^m} \varphi(t) \big\langle \alpha_i(t) \mathcal{T}_q^i u(t) \big\rangle^{(q)}\,\mathrm{d}y \tag{38}$$

(we have used the equality $\langle \varphi(t)\rangle^{(q')} = \varphi(t)$ which is valid since $\mathcal{T}_{q'}\varphi = 0$). Combining (37), (38) yields

$$\int_0^T \int_{\mathbb{R}^m} \left(\partial_t u + \sum_{i=1}^{r} \big\langle \alpha_i \mathcal{T}_q^i u \big\rangle^{(q)} - f \right) \varphi\,\mathrm{d}y\,\mathrm{d}t = 0.$$

Observe that the function $\partial_t u + \sum_{i=1}^{r} \langle \alpha_i \mathcal{T}_q^i u \rangle^{(q)} - f$ belongs to $\ker \mathcal{T}_q$ and thus we obtain

$$\partial_t u + \sum_{i=1}^{r} \big\langle \alpha_i \mathcal{T}_q^i u(t) \big\rangle^{(q)} = f(t), \quad t \in (0, T).$$

Since for any $i \in \{1, \ldots, r\}$ we have $u(t) \in \ker \mathcal{T}_q \cap \mathrm{D}(\mathcal{T}_q^i)$, we deduce by Remark 2.7 that $\mathcal{T}_q^i u(t) \in \ker \mathcal{T}_q$. Therefore, by Corollary 2.4 we obtain

$$\big\langle \alpha_i(t) \mathcal{T}_q^i u(t) \big\rangle^{(q)} = \big\langle \alpha_i(t) \big\rangle^{(\infty)} \mathcal{T}_q^i u(t).$$

Finally u solves

$$\begin{cases} \partial_t u + \sum_{i=1}^{r} \big\langle \alpha_i(t) \big\rangle^{(\infty)} \mathcal{T}_q^i u(t) = f(t), \quad t \in (0, T), \\ u(0) = u_0. \end{cases} \qquad \square \tag{39}$$

Remark 3.1. Notice that if u is a strong solution of (39) whose initial condition belongs to $\ker \mathcal{T}_q$ then the constraint $\mathcal{T}_q u = 0$ is automatically satisfied. Indeed, we have

$$\sum_{i=1}^{r} \big\langle \alpha_i(t) \big\rangle^{(\infty)} \mathcal{T}_q^i u(t) \in \ker \mathcal{T}_q, \quad t \in [0, T],$$

and therefore $\partial_t u \in \ker \mathcal{T}_q$. We deduce that $\partial_t \mathcal{T}_q u = 0$ implying that $\mathcal{T}_q u(t) = \mathcal{T}_q u_0 = 0$ for $t \in [0, T]$.

Remark 3.2. It is easily seen that any strong solution of (28) is also weak solution for the same problem.

Remark 3.3. The strong formulation is a transport problem corresponding to the averaged advection field $\sum_{i=1}^{r} \langle \alpha_i(t) \rangle^{(\infty)} b^i$ and thus very easy to solve numerically.

As usual, the existence of strong solution for the adjoint problem implies the uniqueness of weak solution.

Proposition 3.3. *Assume that* (32), (33) *hold. Then for any* $u_0 \in \ker \mathcal{T}_q$ *and* $f \in L^1([0, T]; \ker \mathcal{T}_q)$, *with* $q \in (1, +\infty)$, *there is at most one weak solution of* (28).

Proof. Let $u \in L^\infty([0,T];\ker \mathcal{T}_q)$ be any weak solution of (28) with vanishing initial condition and source term. We will show that $u = 0$. We know that

$$\int_0^T \int_{\mathbb{R}^m} u(\partial_t \theta + a \cdot \nabla_y \theta)\,\mathrm{d}y\,\mathrm{d}t = 0 \tag{40}$$

for any function $\theta \in C_c^1([0,T) \times \mathbb{R}^m)$ satisfying $\mathcal{T}\theta = 0$. Consider a function $\eta = \eta(t) \in C([0,T])$ and $\psi = \psi(y) \in (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_{q'}^i)) \cap \ker \mathcal{T}_{q'}$. By Corollary 3.1 there is a strong solution $\tilde{\varphi}$ of

$$\begin{cases} \partial_t \tilde{\varphi} - \langle a(T-t) \cdot \nabla_y \tilde{\varphi}\rangle^{(q')} = \eta(T-t)\psi(y), & (t,y) \in (0,T) \times \mathbb{R}^m, \\ \tilde{\varphi}(0,y) = 0, & y \in \mathbb{R}^m, \end{cases}$$

satisfying $\tilde{\varphi}, \mathcal{T}_{q'}^i \tilde{\varphi} \in L^\infty([0,T];L^{q'}(\mathbb{R}^m))$, $\partial_t \tilde{\varphi} \in L^1([0,T];L^{q'}(\mathbb{R}^m))$. It is easily seen that $\varphi(t,y) = \tilde{\varphi}(T-t,y)$ has the same regularity as $\tilde{\varphi}$, $\varphi(t) \in \ker \mathcal{T}_{q'}$ and

$$\begin{cases} -\partial_t \varphi - \langle a(t) \cdot \nabla_y \varphi\rangle^{(q')} = \eta(t)\psi(y), & (t,y) \in (0,T) \times \mathbb{R}^m, \\ \varphi(T,y) = 0, & y \in \mathbb{R}^m. \end{cases}$$

We use now (40) with the function φ (observe that the formulation (40) still holds true for trial functions having the regularity of φ)

$$\begin{aligned} 0 &= \int_0^T \int_{\mathbb{R}^m} u(\partial_t \varphi + a \cdot \nabla_y \varphi)\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^m} u \partial_t \varphi\,\mathrm{d}y\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^m} \langle u(t)\rangle^{(q)} a \cdot \nabla_y \varphi\,\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^m} u\big(\partial_t \varphi + \langle a(t) \cdot \nabla_y \varphi\rangle^{(q')}\big)\,\mathrm{d}y\,\mathrm{d}t \\ &= -\int_0^T \eta(t) \int_{\mathbb{R}^m} u(t,y)\psi(y)\,\mathrm{d}y\,\mathrm{d}t. \end{aligned}$$

We deduce that $\int_{\mathbb{R}^m} u(t,y)\psi(y)\,\mathrm{d}y = 0$ for any $t \in [0,T]$ and any $\psi \in (\bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_{q'}^i)) \cap \ker \mathcal{T}_{q'}$. Since $u(t) \in \ker \mathcal{T}_q$ it follows that $u(t) = 0$, $t \in [0,T]$. $\square$

Remark 3.4. The uniqueness of the weak solution for (28) guarantees the uniqueness of the strong solution in Corollary 3.1.

For further use we establish the conservation of the L^q norm for weak solutions without source term. We need the easy lemma.

Lemma 3.1. *Let $\beta \in W^{1,\infty}(\mathbb{R}^m)$ be a smooth function and $c(y)$ a smooth field with bounded divergence. Assume that $v \in \mathrm{D}(c \cdot \nabla_y) \subset L^q(\mathbb{R}^m)$ for some $q \in (1,+\infty)$. Then we have*

$$\int_{\mathbb{R}^m} \beta(y)(c\cdot\nabla_y)v|v|^{q-2}v\,\mathrm{d}y = -\frac{1}{q}\int_{\mathbb{R}^m} |v|^q \operatorname{div}_y(\beta c)\,\mathrm{d}y.$$

Corollary 3.2. *Assume that* (32), (33) *hold and that* $u_0 \in \ker \mathcal{T}_q$, $f \in L^1([0,T]; \ker \mathcal{T}_q)$ *for some* $q \in (1,+\infty)$. *Then the weak solution of* (28) *satisfies for any* $t \in [0,T]$

$$\frac{1}{q}\int_{\mathbb{R}^m} \left|u(t,y)\right|^q \mathrm{d}y = \frac{1}{q}\int_{\mathbb{R}^m} \left|u_0(y)\right|^q \mathrm{d}y + \int_0^t \int_{\mathbb{R}^m} f(s,y)\left|u(s,y)\right|^{q-2} u(s,y)\,\mathrm{d}y\,\mathrm{d}s.$$

In particular, when $f = 0$ *the* L^q *norm is preserved.*

Proof. Consider the sequences of smooth functions $(u_{0n})_n$ and $(f_n)_n$ such that $\lim_{n\to+\infty} u_{0n} = u_0$ in $L^q(\mathbb{R}^m)$, $\lim_{n\to+\infty} f_n = f$ in $L^1([0,T]; L^q(\mathbb{R}^m))$. Let us denote by u, u_n the unique solutions associated to (u_0, f), (u_{0n}, f_n) respectively. Thanks to the uniqueness result of Proposition 3.3 we deduce by Proposition 3.1 that

$$\|u_n - u\|_{L^\infty([0,T]; L^q(\mathbb{R}^m))} \leqslant \|u_{0n} - u_0\|_{L^q(\mathbb{R}^m)} + \|f_n - f\|_{L^1([0,T]; L^q(\mathbb{R}^m))}$$

and therefore it is sufficient to analyze the case of strong solutions $(u_n)_n$. Taking into account that $|u_n|^{q-2}u_n \in \ker \mathcal{T}_{q'}$ we have by Lemma 3.1

$$\begin{aligned}
\int_{\mathbb{R}^m} \left\langle a(t)\cdot\nabla_y u_n(t)\right\rangle^{(q)} |u_n|^{q-2}u_n\,\mathrm{d}y &= \int_{\mathbb{R}^m} a(t)\cdot\nabla_y u_n(t)\left\langle \left|u_n(t)\right|^{q-2} u_n(t)\right\rangle^{(q')} \mathrm{d}y \\
&= \int_{\mathbb{R}^m} a(t)\cdot\nabla_y u_n(t)\left|u_n(t)\right|^{q-2} u_n(t)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^m} a(t)\cdot\nabla_y \frac{|u_n(t)|^q}{q}\,\mathrm{d}y \\
&= -\frac{1}{q}\int_{\mathbb{R}^m} |u_n|^q \operatorname{div}_y a\,\mathrm{d}y = 0.
\end{aligned}$$

Our conclusion follows immediately by multiplying the equation $\partial_t u_n + \langle a(t)\cdot\nabla_y u_n(t)\rangle^{(q)} = f_n(t)$ by $|u_n(t)|^{q-2}u_n(t)$ and integrating with respect to $y \in \mathbb{R}^m$. $\square$

Naturally we can obtain more smoothness for the solution provided that the data are more regular. We present here a simplified version for the homogeneous problem. The proof is a direct consequence of Propositions 3.2, 2.11 and follows by taking the directional derivatives $b^i\cdot\nabla_y$ to the problem (28) (with $f = 0$). The proof is left to the reader.

Proposition 3.4. *Assume that* (32), (33) *hold and let us denote by* u *the solution of* (28) *with* $f = 0$ *and the initial condition* u_0 *satisfying for some* $q \in (1,+\infty)$

$$u_0 \in \left(\bigcap_{i=1}^r \mathrm{D}\big(\mathcal{T}_q^i\big)\right) \cap \ker \mathcal{T}_q, \qquad \mathcal{T}_q^j u_0 \in \bigcap_{i=1}^r \mathrm{D}\big(\mathcal{T}_q^i\big), \quad \forall j \in \{1,\dots,r\}.$$

Then we have

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\left\|\mathcal{T}_q^i\mathcal{T}_q^j u\right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leqslant C\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\left\|\mathcal{T}_q^i\mathcal{T}_q^j u_0\right\|_{L^q(\mathbb{R}^m)} + \sum_{i=1}^{r}\left\|\mathcal{T}_q^i u_0\right\|_{L^q(\mathbb{R}^m)}\right)$$

with C depending on $\sum_{1\leqslant i,j,k\leqslant r}\|\mathcal{T}_q^i\mathcal{T}_q^j\alpha_k\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$, $\sum_{1\leqslant i,j\leqslant r}\|\mathcal{T}_q^i\alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$ *and*

$$\left\|\partial_t^2 u\right\|_{L^1([0,T];L^q)} + \sum_{i=1}^{r}\left\|\partial_t\mathcal{T}_q^i u\right\|_{L^1([0,T];L^q)} \leqslant C\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\left\|\mathcal{T}_q^i\mathcal{T}_q^j u_0\right\|_{L^q} + \sum_{i=1}^{r}\left\|\mathcal{T}_q^i u_0\right\|_{L^q}\right)$$

with C depending on

$$\sum_{1\leqslant i,j,k\leqslant r}\left\|\mathcal{T}_q^i\mathcal{T}_q^j\alpha_k\right\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}, \qquad \sum_{1\leqslant i,j\leqslant r}\left\|\mathcal{T}_q^i\alpha_j\right\|_{L^1([0,T];L^\infty(\mathbb{R}^m))},$$
$$\sum_{i=1}^{r}\|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} \quad \textit{and} \quad \sum_{i=1}^{r}\|\partial_t\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}.$$

3.1. The limit model in terms of prime integrals

As seen before, the limit model for transport equations like (1) is given by

$$\partial_t u + \left\langle a(t)\cdot\nabla_y u(t)\right\rangle = 0, \quad (t,y)\in(0,T)\times\mathbb{R}^m.$$

When the field a is a linear combination of fields in involution with b, the above limit model can be reduced to a transport equation. Moreover the computations simplify when the prime integrals are employed. We detail here this approach, based on prime integral concept. We assume that there are $m-1$ prime integrals, independent of $\mathbb{R}^m$, associated to the field b

$$b\cdot\nabla_y\psi^i = 0, \quad i\in\{1,\dots,m-1\}, \tag{41}$$

$$\operatorname{rank}\left(\frac{\partial\psi^i}{\partial y_j}(y)\right)_{(m-1)\times m} = m-1, \quad y\in\mathbb{R}^m. \tag{42}$$

Let us recall, that generally, around any non-singular point y_0 of b (*i.e.*, $b(y_0)\neq 0$) there are $(m-1)$ independent prime integrals, defined only locally, in a small enough neighborhood of y_0 (see [3, p. 95]). For any $y\in\mathbb{R}^m$ we denote by $M(y)$ the matrix whose lines are $\nabla_y\psi^1,\dots,\nabla_y\psi^{m-1}$ and b. The hypotheses (41), (42) imply that $\det M(y)\neq 0$ for any $y\in\mathbb{R}^m$. The idea is to search for fields $c=c(y)$ such that $c(y)\cdot\nabla_y u$ remains constant along the flow of b for any function u which is constant along the same flow. If u is constant on the characteristics of b, there is a function $v=v(z):\mathbb{R}^{m-1}\to\mathbb{R}$ such that

$$u(y) = v\left(\psi^1(y),\dots,\psi^{m-1}(y)\right), \quad y\in\mathbb{R}^m.$$

Therefore one gets

$$\frac{\partial u}{\partial y_j} = \sum_{k=1}^{m-1}\frac{\partial v}{\partial z_k}\left(\psi^1(y),\dots,\psi^{m-1}(y)\right)\frac{\partial\psi^k}{\partial y_j}$$

implying that

$$c \cdot \nabla_y u = \sum_{k=1}^{m-1} \frac{\partial v}{\partial z_k}\big(\psi^1(y), \dots, \psi^{m-1}(y)\big) \sum_{j=1}^{m} \frac{\partial \psi^k}{\partial y_j} c_j = (\nabla_z v)\big(\psi(y)\big) \cdot \frac{\partial \psi}{\partial y} c(y).$$

In particular, if $\frac{\partial \psi}{\partial y} c(y)$ do not depend on y, the directional derivative $c \cdot \nabla_y u$ remains constant along the trajectories of b. For any $i \in \{1, \dots, m-1\}$ let us denote by $c^i(y)$ the unique solution of the linear system

$$M(y) c^i(y) = e^i := (\delta_{ij})_{1 \leqslant j \leqslant m}$$

where δ_{ij} are the Kronecker's symbols. Notice that $M(y) \frac{b(y)}{|b(y)|^2} = e^m$ and thus the vectors $c^1(y), \dots, c^{m-1}(y), b(y)$ are linearly independent at any $y \in \mathbb{R}^m$. According to the previous computations, for any function u constant along the flow of b, the directional derivative $c^i \cdot \nabla_y u$ remains constant along the same flow for any $i \in \{1, \dots, m-1\}$. We denote by $\beta_0, \beta_1, \dots, \beta_{m-1}$ the coordinates of a with respect to $b, c^1, \dots, c^{m-1}$ and we assume that $(\beta_i)_i$ are smooth and bounded

$$a(t, y) = \beta_0(t, y) b(y) + \sum_{i=1}^{m-1} \beta_i(t, y) c^i(y), \quad (t, y) \in [0, T] \times \mathbb{R}^m. \tag{43}$$

Thanks to Corollary 2.4, one gets for any function $u \in (\bigcap_{i=1}^{m-1} \mathrm{D}(\mathcal{T}_q^{c^i})) \cap \ker \mathcal{T}_q$

$$\big\langle a(t) \cdot \nabla_y u(t) \big\rangle^{(q)} = \left\langle \sum_{i=1}^{m-1} \beta_i(t) c^i(y) \cdot \nabla_y u(t) \right\rangle^{(q)} = \sum_{i=1}^{m-1} \big\langle \beta_i(t) \big\rangle^{(\infty)} c^i(y) \cdot \nabla_y u(t).$$

It remains to compute $(\beta_i)_i$. Multiplying (43) by $M(y)$ yields

$$M(y) a(t, y) = \beta_0(t, y) \big|b(y)\big|^2 e^m + \sum_{i=1}^{m-1} \beta_i(t, y) e^i$$

implying that

$$\beta_i(t, y) = M(y) a(t, y) \cdot e^i, \quad i \in \{1, \dots, m-1\}, \qquad \beta_0(t, y) \big|b(y)\big|^2 = M(y) a(t, y) \cdot e^m$$

or equivalently to

$$\beta_i(t, y) = a(t, y) \cdot \nabla_y \psi^i, \quad i \in \{1, \dots, m-1\}, \qquad \beta_0(t, y) = \frac{a(t, y) \cdot b(y)}{|b(y)|^2}.$$

Finally one gets the following form of the limit model

$$\partial_t u + \sum_{i=1}^{m-1} \big\langle a(t) \cdot \nabla_y \psi^i \big\rangle^{(\infty)} M^{-1}(y) e^i \cdot \nabla_y u = 0. \tag{44}$$

4. Convergence towards the limit model

This section is devoted to the asymptotic behavior of the solutions $(u^\varepsilon)_{\varepsilon>0}$ of (1). We assume that b, a satisfy the hypotheses (10)–(12), (32) and we work in the $L^2(\mathbb{R}^m)$ setting ($q = 2$). Motivated by Hilbert's expansion method, we intend to show the convergence of $(u^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \searrow 0$ towards the solution u of (27). As usual such kind of result is available provided that the solution of the limit model has enough regularity. Therefore we assume that (27) has strong solution. Our main convergence result is the following.

Theorem 4.1. *Assume that $(\alpha_i)_{i\in\{1,\dots,r\}}$ are smooth (let us say $C^2([0,T]\times\mathbb{R}^m)$) and satisfy*

$$\sum_{i=1}^r \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} + \sum_{i=1}^r \|\partial_t\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} < +\infty,$$
$$\sum_{i=1}^r\sum_{j=1}^r \big\|\mathcal{T}_2^i\alpha_j\big\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} + \sum_{i=1}^r\sum_{j=1}^r\sum_{k=1}^r \big\|\mathcal{T}_2^i\mathcal{T}_2^j\alpha_k\big\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} < +\infty.$$

Suppose that

$$u_0 \in \left(\bigcap_{i=1}^r \mathrm{D}\big(\mathcal{T}_2^i\big)\right)\cap \ker\mathcal{T}_2, \qquad \mathcal{T}_2^j u_0 \in \bigcap_{i=1}^r \mathrm{D}\big(\mathcal{T}_2^i\big), \quad \forall j\in\{1,\dots,r\},$$

and that $(u_0^\varepsilon)_{\varepsilon>0}$ are smooth initial conditions (let us say $C^1(\mathbb{R}^m)$) such that $\lim_{\varepsilon\searrow 0} u_0^\varepsilon = u_0$ in $L^2(\mathbb{R}^m)$. We denote by u^ε, u the solutions of (1), (27) respectively. Then we have $\lim_{\varepsilon\searrow 0} u^\varepsilon = u$, in $L^\infty([0,T];L^2(\mathbb{R}^m))$.

Proof. By Propositions 3.2, 3.3 and Corollary 3.2 there is a unique strong solution u for (27), satisfying $\|u(t)\|_{L^2(\mathbb{R}^m)} = \|u_0\|_{L^2(\mathbb{R}^m)}$ for any $t\in[0,T]$ and

$$\|\partial_t u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + \sum_{i=1}^r \big\|\mathcal{T}_2^i u\big\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} \leqslant C\sum_{i=1}^r \big\|\mathcal{T}_2^i u_0\big\|_{L^2(\mathbb{R}^m)}.$$

Since $u(t)\in\ker\mathcal{T}_2$, $t\in[0,T]$, we have

$$\big\langle \partial_t u + a(t)\cdot\nabla_y u(t)\big\rangle^{(2)} = \partial_t\langle u\rangle^{(2)} + \big\langle a(t)\cdot\nabla_y u(t)\big\rangle^{(2)} = \partial_t u + \big\langle a(t)\cdot\nabla_y u(t)\big\rangle^{(2)} = 0$$

and thus by Proposition 2.9 there are $(v_\mu)_{\mu>0}$ such that

$$\begin{aligned} &\partial_t u + a(t,y)\cdot\nabla_y u + \mu v_\mu(t,y) + \mathcal{T}_2 v_\mu = 0,\\ &\lim_{\mu\searrow 0}\big(\mu v_\mu(t)\big) = 0 \quad \text{in } L^2\big(\mathbb{R}^m\big),\ t\in[0,T]. \end{aligned} \tag{45}$$

Moreover, by Remark 2.4 we know that

$$\begin{aligned} \|\mu v_\mu\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} &\leqslant \big\|\partial_t u + a(t)\cdot\nabla_y u\big\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\ &\leqslant \|\partial_t u\|_{L^\infty([0,T];L^2(\mathbb{R}^m))} + C\sum_{i=1}^r \|\alpha_i\|_{W^{1,1}([0,T];L^\infty(\mathbb{R}^m))}\big\|\mathcal{T}_2^i u\big\|_{L^\infty([0,T];L^2(\mathbb{R}^m))}\\ &\leqslant C\sum_{i=1}^r \big\|\mathcal{T}_2^i u_0\big\|_{L^2(\mathbb{R}^m)}. \end{aligned} \tag{46}$$

Combining (1), (27) and the equation $\mathcal{T}_2 u = 0$ yields

$$\left(\partial_t + a(t,y)\cdot\nabla_y + \frac{b(y)}{\varepsilon}\cdot\nabla_y\right)\left(u^\varepsilon - u - \varepsilon v_\mu\right) = \mu v_\mu - \varepsilon\left(\partial_t v_\mu + a(t,y)\cdot\nabla_y v_\mu\right). \tag{47}$$

We investigate now the regularity of v_μ. By Remark 2.4 we have

$$\mu\left\|\partial_t v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} \leqslant \left\|\partial_t^2 u + \sum_{i=1}^r \partial_t \alpha_i \mathcal{T}_2^i u + \sum_{i=1}^r \alpha_i(t)\partial_t \mathcal{T}_2^i u\right\|_{L^2(\mathbb{R}^m)}$$

and thus Proposition 3.4 implies

$$\mu\|\partial_t v_\mu\|_{L^1([0,T];L^2(\mathbb{R}^m))} \leqslant C\left(\sum_{i=1}^r\sum_{j=1}^r \left\|\mathcal{T}_2^i\mathcal{T}_2^j u_0\right\|_{L^2(\mathbb{R}^m)} + \sum_{i=1}^r \left\|\mathcal{T}_2^i u_0\right\|_{L^2(\mathbb{R}^m)}\right). \tag{48}$$

Applying now the operator $\mathcal{T}_2^i$, $i \in \{0, 1, \dots, r\}$, in (45), yields

$$\partial_t \mathcal{T}_2^i u + \sum_{j=1}^r \left\{\left(\mathcal{T}_2^i \alpha_j\right)\left(\mathcal{T}_2^j u\right) + \alpha_j\left(\mathcal{T}_2^i \mathcal{T}_2^j u\right)\right\} + \mu \mathcal{T}_2^i v_\mu + \mathcal{T}_2 \mathcal{T}_2^i v_\mu = 0.$$

By Remark 2.4 and Proposition 3.4 we obtain as before

$$\mu\left\|\mathcal{T}_2^i v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} \leqslant \left\|\partial_t \mathcal{T}_2^i u(t) + \sum_{j=1}^r \left\{\left(\mathcal{T}_2^i \alpha_j(t)\right)\left(\mathcal{T}_2^j u(t)\right) + \alpha_j(t)\left(\mathcal{T}_2^i \mathcal{T}_2^j u(t)\right)\right\}\right\|_{L^2(\mathbb{R}^m)}$$

implying that

$$\mu\sum_{i=0}^r \left\|\mathcal{T}_2^i v_\mu\right\|_{L^1([0,T];L^2(\mathbb{R}^m))} \leqslant C\left(\sum_{i=1}^r\sum_{j=1}^r \left\|\mathcal{T}_2^i\mathcal{T}_2^j u_0\right\|_{L^2(\mathbb{R}^m)} + \sum_{i=1}^r \left\|\mathcal{T}_2^i u_0\right\|_{L^2(\mathbb{R}^m)}\right). \tag{49}$$

Multiplying (47) by $u^\varepsilon - u - \varepsilon v_\mu$ and integrating over $\mathbb{R}^m$ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\left(u^\varepsilon - u - \varepsilon v_\mu\right)(t)\right\|_{L^2(\mathbb{R}^m)}^2 \leqslant{}& \left\|\mu v_\mu(t)\right\|_{L^2(\mathbb{R}^m)}\left\|\left(u^\varepsilon - u - \varepsilon v_\mu\right)(t)\right\|_{L^2(\mathbb{R}^m)} \\ &+ \varepsilon\left\|\partial_t v_\mu(t) + \sum_{i=0}^r \alpha_i(t)\mathcal{T}_2^i v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} \\ &\times \left\|\left(u^\varepsilon - u - \varepsilon v_\mu\right)(t)\right\|_{L^2(\mathbb{R}^m)}\end{aligned}$$

and we deduce that

$$\frac{d}{dt}\left\|\left(u^\varepsilon - u - \varepsilon v_\mu\right)(t)\right\|_{L^2(\mathbb{R}^m)} \leqslant \left\|\mu v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} + C\varepsilon\left(\left\|\partial_t v_\mu(t)\right\|_{L^2(\mathbb{R}^m)} + \sum_{i=0}^r \left\|\mathcal{T}_2^i v_\mu(t)\right\|_{L^2(\mathbb{R}^m)}\right).$$

Combining with (48), (49), we obtain for any $t \in [0, T]$

$$\begin{aligned}\left\|\left(u^{\varepsilon}-u-\varepsilon v_{\mu}\right)(t)\right\|_{L^2(\mathbb{R}^m)} &\leqslant \left\|u_0^{\varepsilon}-u_0-\varepsilon v_{\mu}(0)\right\|_{L^2(\mathbb{R}^m)}+\int_0^T \left\|\mu v_{\mu}(s)\right\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}s \\ &\quad + C\frac{\varepsilon}{\mu}\left(\|\mu\partial_t v_{\mu}\|_{L^1([0,T];L^2(\mathbb{R}^m))}+\sum_{i=0}^{r}\left\|\mu\mathcal{T}_2^i v_{\mu}\right\|_{L^1([0,T];L^2(\mathbb{R}^m))}\right) \\ &\leqslant \left\|u_0^{\varepsilon}-u_0-\varepsilon v_{\mu}(0)\right\|_{L^2(\mathbb{R}^m)}+\int_0^T \left\|\mu v_{\mu}(s)\right\|_{L^2(\mathbb{R}^m)}\,\mathrm{d}s + C\frac{\varepsilon}{\mu}.\end{aligned}$$

Consequently one gets by (46) for any $t \in [0, T]$

$$\begin{aligned}\left\|\left(u^{\varepsilon}-u\right)(t)\right\|_{L^2(\mathbb{R}^m)} &\leqslant \left\|u_0^{\varepsilon}-u_0\right\|_{L^2(\mathbb{R}^m)}+\frac{\varepsilon}{\mu}\left(\left\|\mu v_{\mu}(t)\right\|_{L^2(\mathbb{R}^m)}+\left\|\mu v_{\mu}(0)\right\|_{L^2(\mathbb{R}^m)}\right) \\ &\quad + C\frac{\varepsilon}{\mu}+\|\mu v_{\mu}\|_{L^1([0,T];L^2(\mathbb{R}^m))} \\ &\leqslant \left\|u_0^{\varepsilon}-u_0\right\|_{L^2(\mathbb{R}^m)}+C\frac{\varepsilon}{\mu}+\|\mu v_{\mu}\|_{L^1([0,T];L^2(\mathbb{R}^m))}.\end{aligned}$$

Since the functions $t \to \|\mu v_{\mu}(t)\|_{L^2(\mathbb{R}^m)}$ converge pointwise to 0 as $\mu \searrow 0$ (cf. (45)) and they are uniformly bounded on $[0, T]$ (cf. (46)) we deduce by dominated convergence theorem that

$$\lim_{\mu\searrow 0}\|\mu v_{\mu}\|_{L^1([0,T];L^2(\mathbb{R}^m))}=0.$$

In particular, for $\mu = \varepsilon^{\delta}$, with $\delta \in (0, 1)$ we have

$$\begin{aligned}&\left\|u^{\varepsilon}-u\right\|_{L^{\infty}([0,T];L^2(\mathbb{R}^m))} \\ &\quad\leqslant \left\|u_0^{\varepsilon}-u_0\right\|_{L^2(\mathbb{R}^m)}+C\varepsilon^{1-\delta}+\left\|\varepsilon^{\delta}v_{\varepsilon^{\delta}}\right\|_{L^1([0,T];L^2(\mathbb{R}^m))}\to 0, \quad \text{as } \varepsilon\searrow 0. \qquad \square\end{aligned}$$

5. Applications

In this section we present some applications of the results obtained before. It mainly concerns the effects of strong magnetic fields. Nevertheless, the method applies in many other situations, each time we need to separate multiple scales. Motivated by the magnetic confinement fusion, which is one of the main applications in plasma physics today, we chose to analyze the dynamics of a population of charged particles (electrons) under the action of strong magnetic fields $B^{\varepsilon} = B/\varepsilon$, $0 < \varepsilon \ll 1$. Using the kinetic description, the evolution of the particle population is given in terms of a probability density $f = f(t, x, p) \geqslant 0$ depending on time t, position x, momentum p. When neglecting the collisions this particle density satisfies the Vlasov equation

$$\partial_t f+\frac{p}{m_{\mathrm{e}}}\cdot\nabla_x f-e\left(E(t,x)+\frac{p}{m_{\mathrm{e}}}\wedge B^{\varepsilon}(t,x)\right)\cdot\nabla_p f=0$$

where $-e < 0$ is the electron charge and $m_{\mathrm{e}} > 0$ is the electron mass. The time evolution of the electro-magnetic field (E, B) comes by the Maxwell equations. For simplicity we restrict ourselves to the linear Vlasov equation by considering that $B = B(x)$ is a given stationary external magnetic field and that the electric field derives from a potential $E = \nabla_x\phi$. The asymptotic regimes we wish to address here are the guiding-center approximation and the finite Larmor radius regime. Certainly, these regimes are now well understood, cf. [5,9,11,10]. Nevertheless our approach allows us to analyze both models by the same method, to treat more general situations, to compute the drift velocities in the orthogonal directions with respect to the magnetic field, etc.

The numerical approximation of the gyrokinetic models has been performed in [12] using semi-Lagrangian schemes. Other methods are based on the water bag representation of the distribution function: the full kinetic Vlasov equation is reduced to a set of hydrodynamic equations. This technique has been successfully applied to gyrokinetic models [13]. We also mention that the drift approximation of strongly magnetized plasmas is analogous to the geostrophic flow in the theory of a shallow rotating fluid [1,2,7,16,17].

We consider here only the two-dimensional setting, *i.e.*,

$$f = f(t, x, p), \qquad (E, B) = \big(E_1(t, x), E_2(t, x), 0, 0, 0, B_3(x)\big), \quad (t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2,$$

leading to the Vlasov equation

$$\partial_t f^\varepsilon + \frac{p}{m_e} \cdot \nabla_x f^\varepsilon - e\left(E(t, x) + \frac{B_3(x)}{\varepsilon} \frac{{}^\perp p}{m_e}\right) \cdot \nabla_p f^\varepsilon = 0 \tag{50}$$

where the notation ${}^\perp p$ stands for $(p_2, -p_1)$ for any $p = (p_1, p_2) \in \mathbb{R}^2$. We only indicate the main steps but we clearly identify the average operators involved in the analysis. The reader can easily adapt to the Vlasov equation the rigorous arguments detailed in the general linear transport framework in order to justify the asymptotic behavior towards the limit model (cf. Theorem 4.1). We concentrate on the derivation of these limit models by applying the properties of the average operators.

5.1. Guiding-center approximation

The asymptotic regime obtained for $\varepsilon \searrow 0$ in (50) is known as the guiding-center approximation, since the Larmor radius corresponding to the typical momentum vanishes as the magnetic field becomes very large. The Vlasov equation (50) can be written

$$\partial_t f^\varepsilon + \mathcal{A} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{T} f^\varepsilon = 0, \quad (t, x, p) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2, \tag{51}$$

where $\mathcal{A} = \frac{p}{m_e} \cdot \nabla_x - eE(t, x) \cdot \nabla_p$ and $\mathcal{T} = -\omega_c(x)^\perp p \cdot \nabla_p$. Here $\omega_c(x) = \frac{eB_3(x)}{m_e}$ stands for the (rescaled) cyclotronic frequency. We complete the above model with the initial condition

$$f^\varepsilon(0, x, p) = f^{\text{in}}(x, p), \quad (x, p) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{52}$$

Notice that (51) can be recast in the form (1) by taking $m = 4$, $y = (x, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $u^\varepsilon(t, y) = f^\varepsilon(t, x, p)$, $a(t, y) = (\frac{p}{m_e}, -eE(t, x))$, $b(y) = (0, 0, -eB_3(x)\frac{{}^\perp p}{m_e})$. It is easily seen that the characteristic flow associated to the (dominant) transport operator $\mathcal{T}$ is given by

$$X(s; x, p) = x, \qquad P(s; x, p) = R\big(\omega_c(x)s\big)p, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The motion is $T_c(x) = \frac{2\pi}{|\omega_c(x)|}$ periodic and thus the average operator has the form

$$\langle u\rangle(x,p)=\frac{1}{T_c(x)}\int_0^{T_c(x)} u\big(X(s;x,p),P(s;x,p)\big)\,\mathrm{d}s$$
$$=\frac{1}{2\pi}\int_{S^1} u\big(x,|p|\omega\big)\,\mathrm{d}\omega. \tag{53}$$

The operators $\mathcal{A},\mathcal{T},\langle\cdot\rangle$ satisfy

Lemma 5.1.

i) *If* $\inf_{x\in\mathbb{R}^2}|B_3(x)|>0$ *then the range of* $\mathcal{T}$ *is closed and we have* $\operatorname{range}\mathcal{T}=\ker\langle\cdot\rangle$.
ii) *For any* $f\in\ker\mathcal{T}$ *then* $\mathcal{A}f\in\operatorname{range}\mathcal{T}$ *and* $\mathcal{A}f=\frac{1}{\omega_c(x)}\mathcal{T}(\frac{^\perp p}{m_e}\cdot\nabla_x f+e^\perp E(t)\cdot\nabla_p f)$.

Proof. The first statement follows by Proposition 2.8 with $T=\frac{2\pi}{\omega}$, $\omega=\frac{|e|}{m_e}\inf_{x\in\mathbb{R}^2}|B_3(x)|$. For the second one observe that for any $u=(u_1,u_2)\in(\ker\mathcal{T})^2$ we have $(^\perp p\cdot\nabla_p)(p\cdot u)={}^\perp p\cdot u$. Notice also that if $f\in\ker\mathcal{T}$ then $\nabla_x f\in(\ker\mathcal{T})^2$. Consequently one gets

$$\frac{p}{m_e}\cdot\nabla_x f=\frac{^\perp p}{m_e}\cdot{}^\perp\nabla_x f=\big(^\perp p\cdot\nabla_p\big)\bigg(\frac{p}{m_e}\cdot{}^\perp\nabla_x f\bigg).$$

In order to transform the second term in $\mathcal{A}$ observe that any function in $\ker\mathcal{T}$ is radial with respect to p, *i.e.*, $f(x,p)=g(x,r=|p|)$. Therefore $eE(t)\cdot\nabla_p f=(^\perp p\cdot\nabla_p)(e^\perp E(t)\cdot\nabla_p f)$ and finally we obtain $\omega_c(x)\mathcal{A}f=\mathcal{T}(\frac{^\perp p}{m_e}\cdot\nabla_x f+e^\perp E(t)\cdot\nabla_p f)$. □

Remark 5.1. A straightforward computation shows that for any $f\in\ker\mathcal{T}$ we have

$$\left\langle\frac{^\perp p}{m_e}\cdot\nabla_x f\right\rangle=\big\langle e^\perp E(t)\cdot\nabla_p f\big\rangle=0.$$

Plugging the ansatz $f^\varepsilon=f+\varepsilon f^1+\varepsilon^2 f^2+\cdots$ into (51) we deduce that

$$\mathcal{T}f=0,\qquad \partial_t f+\mathcal{A}f+\mathcal{T}f^1=0,\qquad \partial_t f^1+\mathcal{A}f^1+\mathcal{T}f^2=0,\ \ldots. \tag{54}$$

In order to identify the limit model satisfied by $f=\lim_{\varepsilon\searrow 0}f^\varepsilon$ we apply, as before, the average operator to the second equation in (54) and one gets $\partial_t f+\langle\mathcal{A}f(t)\rangle=0$. By Lemma 5.1 we know that $\mathcal{A}f(t)\in\operatorname{range}\mathcal{T}$ and thus $\langle\mathcal{A}f(t)\rangle=0$. Finally the dominant term satisfies $\partial_t f=0$. For identifying the initial condition multiply (51) by a test function $\eta(t)\varphi(x,p)$ with $\eta\in C_c^1(\mathbb{R}_+)$ and $\varphi\in C_c^1(\mathbb{R}^2\times\mathbb{R}^2)\cap\ker\mathcal{T}$. Passing to the limit as $\varepsilon\searrow 0$ yields

$$-\int_{\mathbb{R}_+}\eta'(t)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x,p)\varphi(x,p)\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}t-\eta(0)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{\mathrm{in}}(x,p)\varphi(x,p)\,\mathrm{d}p\,\mathrm{d}x=0.$$

Taking into account that $\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^{\mathrm{in}}(x,p)\varphi(x,p)\,\mathrm{d}p\,\mathrm{d}x=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\langle f^{\mathrm{in}}\rangle\varphi(x,p)\,\mathrm{d}p\,\mathrm{d}x$ one gets that $f(t)=\langle f^{\mathrm{in}}\rangle$, $t\in\mathbb{R}_+$. At the lowest order the particle density is stationary and has radial symmetry with respect to p

$$\lim_{\varepsilon\searrow 0}f^\varepsilon(t)=\big\langle f^{\mathrm{in}}\big\rangle,\quad t\in\mathbb{R}_+.$$

Consequently, at this order, there is no current

$$j(t,x) = -e \int_{\mathbb{R}^2} f(t,x,p) \frac{p}{m_e} \, \mathrm{d}p = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2.$$

In the sequel we intend to compute the first order drift velocities which are very important for the analysis of the confinement properties. In order to compute f^1 we use the decomposition $f^1 = g^1 + h^1$, $g^1 = \langle f^1 \rangle$, $h^1 = f^1 - \langle f^1 \rangle$. Notice that $\mathcal{T} g^1 = 0$ and $\langle h^1 \rangle = 0$. The second equation in (54) combined with Lemma 5.1 and Remark 5.1 lead to

$$\frac{1}{\omega_c(x)} \left(\frac{^\perp p}{m_e} \cdot \nabla_x f + e^\perp E(t) \cdot \nabla_p f \right) + h^1 \in \ker \mathcal{T} \cap \ker \langle \cdot \rangle = \{0\}$$

and therefore $h^1(t) = -\frac{1}{\omega_c(x)}(\frac{^\perp p}{m_e} \cdot \nabla_x f + e^\perp E(t) \cdot \nabla_p f)$. For determining g^1 we use the third equation in (54) after eliminating f^2 by applying the average operator. Since $\langle \partial_t h^1 \rangle = \partial_t \langle h^1 \rangle = 0$ and $\langle \partial_t g^1 \rangle = \partial_t \langle g^1 \rangle = \partial_t g^1$, $\langle \mathcal{A} g^1 \rangle = 0$ (by Lemma 5.1) one gets $\partial_t g^1 + \langle \mathcal{A} h^1 \rangle = 0$. Actually the radial symmetric density g^1 gives no current and thus we do not need to compute it explicitly. We have

$$j^1 = -e \int_{\mathbb{R}^2} f^1 \frac{p}{m_e} \, \mathrm{d}p = -e \int_{\mathbb{R}^2} h^1 \frac{p}{m_e} \, \mathrm{d}p = \frac{e}{\omega_c(x) m_e^2} \int_{\mathbb{R}^2} \operatorname{div}_x \big(p \otimes {}^\perp p f \big) \, \mathrm{d}p - \frac{e^{2 \perp} E}{m_e \omega_c(x)} \int_{\mathbb{R}^2} f \, \mathrm{d}p.$$

We introduce the charge density and cyclotronic velocity given by

$$\rho^{\mathrm{in}}(x) = -e \int_{\mathbb{R}^2} f^{\mathrm{in}}(x,p) \, \mathrm{d}p, \qquad \frac{m_e (V_c^{\mathrm{in}}(x))^2}{2} = \frac{\int_{\mathbb{R}^2} \frac{|p|^2}{2 m_e} f^{\mathrm{in}}(x,p) \, \mathrm{d}p}{\int_{\mathbb{R}^2} f^{\mathrm{in}}(x,p) \, \mathrm{d}p}.$$

Notice that we have by symmetry

$$\int_{\mathbb{R}^2} f \, \mathrm{d}p = \int_{\mathbb{R}^2} f^{\mathrm{in}} \, \mathrm{d}p \quad \text{and} \quad \operatorname{div}_x \int_{\mathbb{R}^2} \big(p \otimes {}^\perp p \big) f \, \mathrm{d}p = -{}^\perp \nabla_x \int_{\mathbb{R}^2} \frac{|p|^2}{2} f^{\mathrm{in}} \, \mathrm{d}p.$$

By direct computations one gets

$$j^1(t,x) = {}^\perp \nabla_x \left(\rho^{\mathrm{in}}(x) \frac{(V_c^{\mathrm{in}}(x))^2}{2\omega_c(x)} \right) + \rho^{\mathrm{in}}(x) \frac{(V_c^{\mathrm{in}}(x))^2}{2\omega_c(x)} \frac{^\perp \nabla_x B_3}{B_3(x)} + \rho^{\mathrm{in}}(x) \frac{^\perp E(t,x)}{B_3(x)}.$$

We recognize here the cross electric field drift and the magnetic gradient drift given by the standard formula used by physicists (cf. [14, p. 162])

$$v_\wedge = \frac{^\perp E}{B_3} = \frac{E \wedge B}{|B|^2}, \qquad v_{\mathrm{GD}} = \frac{(V_c^{\mathrm{in}}(x))^2}{2\omega_c(x)} \frac{^\perp \nabla_x B_3}{B_3(x)} = -\frac{(V_c^{\mathrm{in}}(x))^2}{2\omega_c(x)} \frac{B \wedge \nabla_x B_3}{|B|^2}.$$

The previous results are summarized in

Proposition 5.1. *Under the hypothesis in Lemma* 5.1 *we have*

$$f^\varepsilon = \langle f^{\mathrm{in}} \rangle + o(\varepsilon),$$

$$j^\varepsilon := -e \int_{\mathbb{R}^2} f^\varepsilon \frac{p}{m_e}\, \mathrm{d}p = \varepsilon \left[{}^\perp\nabla_x \left(\rho^{\mathrm{in}}(x) \frac{(V_c^{\mathrm{in}}(x))^2}{2\omega_c(x)} \right) + \rho^{\mathrm{in}}(x) v_{\mathrm{GD}} + \rho^{\mathrm{in}}(x) v_\wedge \right] + \varepsilon o(\varepsilon).$$

5.2. Finite Larmor radius regime

In this case we assume that the (scaled) typical momentum in the plane orthogonal to the magnetic field is very large, remaining of the same order as the magnetic field. Note that in this case the Larmor radius corresponding to the typical velocity and cyclotronic frequency doesn't vanish anymore. We obtain the Vlasov equation (see [8,4])

$$\partial_t f^\varepsilon + \frac{p}{m_e \varepsilon} \cdot \nabla_x f^\varepsilon - e\left(E(t,x) + B_3(x) \frac{{}^\perp p}{m_e \varepsilon} \right) \cdot \nabla_p f^\varepsilon = 0 \tag{55}$$

and the corresponding asymptotic regime for $\varepsilon \searrow 0$ is called the finite Larmor radius regime. Observe that (55) can be recast in the form (1) by taking $m = 4$, $y = (x,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $u^\varepsilon(t,y) = f^\varepsilon(t,x,p)$, $\tilde{a}(t,y) = -(0,0,eE(t,x))$, $\tilde{b}(y) = (\frac{p}{m_e}, -\omega_c(x)^\perp p)$, where $\omega_c(x) = eB_3(x)/m_e$. The characteristic flow $Y = (X,P)$ associated to $\tilde{b}$ satisfies

$$\frac{dX}{ds} = \frac{P(s;x,p)}{m_e}, \qquad \frac{dP}{ds} = -\omega_c\big(X(s;x,p)\big)^\perp P(s;x,p).$$

When B_3 is constant it is easily seen that a set of independent prime integrals is given by

$$\tilde{\psi}^1(x,p) = eB_3 x_2 + p_1, \qquad \tilde{\psi}^2(x,p) = -eB_3 x_1 + p_2, \qquad \tilde{\psi}^3(p) = \frac{1}{2}|p|^2.$$

We intend to derive the limit model using the arguments in Section 3.1 and thus we need to invert the matrix

$$\tilde{M}(p) = \begin{pmatrix} 0 & eB_3 & 1 & 0 \\ -eB_3 & 0 & 0 & 1 \\ 0 & 0 & p_1 & p_2 \\ \frac{p_1}{m_e} & \frac{p_2}{m_e} & -\omega_c p_2 & \omega_c p_1 \end{pmatrix}.$$

In order to simplify our computations it is very convenient to introduce the new variable $z = x - \frac{{}^\perp p}{eB_3} = (-\tilde{\psi}^2, \tilde{\psi}^1)/(eB_3)$ and the new unknown $g^\varepsilon(t,z,p) = f^\varepsilon(t,x,p)$. The equation for g^ε becomes

$$\partial_t g^\varepsilon + \frac{1}{B_3} {}^\perp E\left(t, z + \frac{{}^\perp p}{eB_3}\right) \cdot \nabla_z g^\varepsilon - eE\left(t, z + \frac{{}^\perp p}{eB_3}\right) \cdot \nabla_p g^\varepsilon - \frac{\omega_c}{\varepsilon} {}^\perp p \cdot \nabla_p g^\varepsilon = 0$$

and thus the fields to analyze in this case are

$$a(t,z,p) = \left(\frac{1}{B_3} {}^\perp E\left(t, z + \frac{{}^\perp p}{eB_3}\right), -eE\left(t, z + \frac{{}^\perp p}{eB_3}\right) \right), \qquad b(p) = \big(0, 0, -\omega_c {}^\perp p\big).$$

A set of independent prime integrals is given by

$$\psi^1 = z_1, \qquad \psi^2 = z_2, \qquad \psi^3 = \frac{1}{2}|p|^2.$$

The matrix $M(p)$ and its inverse are given by

$$M(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p_1 & p_2 \\ 0 & 0 & -\omega_c p_2 & \omega_c p_1 \end{pmatrix}, \qquad M^{-1}(p) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{p_1}{|p|^2} & -\frac{p_2}{\omega_c |p|^2} \\ 0 & 0 & \frac{p_2}{|p|^2} & \frac{p_1}{\omega_c |p|^2} \end{pmatrix}.$$

In view of (44) we need to compute $\langle a(t) \cdot \nabla_{(z,p)} \psi^i \rangle^{(\infty)}$, $i \in \{1, 2, 3\}$. A direct computation shows that the flow $(Z, P)(s; z, p)$ associated to b is given by

$$Z(s; z, p) = z, \qquad P(s; z, p) = R(s\omega_c)p, \qquad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Consequently the constant functions along the flow are the functions with radial symmetry with respect to p. Observe also that the hypothesis (18) holds true with $\mathcal{O} = \emptyset$ and $\xi(z, p) = \mathrm{e}^{-|z|^2 - |p|^2}$, $(z, p) \in \mathbb{R}^4$. Since all the trajectories are $2\pi/\omega_c$ periodic, we have

$$\langle u \rangle^{(\infty)}(z, p) = \frac{\omega_c}{2\pi} \int_0^{\frac{2\pi}{\omega_c}} u\big(z, R(s\omega_c)p\big)\,\mathrm{d}s = \frac{1}{2\pi} \int_0^{2\pi} u\big(z, R(\theta)p\big)\,\mathrm{d}\theta$$

for any bounded function $u \in L^\infty(\mathbb{R}^4)$. We have

$$\big\langle a(t) \cdot \nabla_{(z,p)} \psi^1 \big\rangle^{(\infty)} = \left\langle \frac{1}{B_3} E_2\left(t, z + \frac{{}^\perp p}{eB_3}\right) \right\rangle^{(\infty)} = \frac{1}{2\pi B_3} \int_0^{2\pi} E_2\left(t, z + \frac{{}^\perp (R(\theta)p)}{eB_3}\right) \mathrm{d}\theta,$$

$$\big\langle a(t) \cdot \nabla_{(z,p)} \psi^2 \big\rangle^{(\infty)} = -\left\langle \frac{1}{B_3} E_1\left(t, z + \frac{{}^\perp p}{eB_3}\right) \right\rangle^{(\infty)} = -\frac{1}{2\pi B_3} \int_0^{2\pi} E_1\left(t, z + \frac{{}^\perp (R(\theta)p)}{eB_3}\right) \mathrm{d}\theta.$$

We claim that the coefficient $\langle a(t) \cdot \nabla_{(z,p)} \psi^3 \rangle^{(\infty)}$ vanishes. Indeed

$$\big\langle a(t) \cdot \nabla_{(z,p)} \psi^3 \big\rangle^{(\infty)} = -\frac{e\omega_c}{2\pi} \int_0^{\frac{2\pi}{\omega_c}} E\left(t, z + \frac{{}^\perp p P(s; z, p)}{eB_3}\right) \cdot P(s; z, p)\,\mathrm{d}s.$$

Taking into account that $E(t)$ derives from a potential $\phi(t)$, *i.e.*, $E = \nabla_x \phi$ and that

$$\frac{d}{ds}\phi\left(t, z + \frac{{}^\perp P(s; z, p)}{eB_3}\right) = E\left(t, z + \frac{{}^\perp P(s; z, p)}{eB_3}\right) \cdot \frac{P(s; z, p)}{m_\mathrm{e}}$$

we deduce that

$$\langle a(t)\cdot\nabla_{(z,p)}\psi^3\rangle^{(\infty)} = -\frac{em_e\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \frac{d}{ds}\phi\left(t, z+\frac{{}^\perp P(s;z,p)}{eB_3}\right)\mathrm{d}s = 0.$$

Plugging into (44) all these computations yields the limit model

$$\partial_t g + \frac{1}{2\pi B_3}\int_0^{2\pi} {}^\perp E\left(t, z+\frac{{}^\perp(R(\theta)p)}{eB_3}\right)\mathrm{d}\theta\cdot\nabla_z g = 0$$

leading to a transport equation for the particle density f, whose advection field is given by a gyroaverage type operator.

Proposition 5.2. *If the magnetic field is constant $B_3 \neq 0$ then the limit model of* (55) *when $\varepsilon \searrow 0$ is given by*

$$\partial_t f + \frac{1}{2\pi B_3}\int_0^{2\pi} {}^\perp E\left(t, x-\frac{{}^\perp p}{eB_3}+\frac{{}^\perp(R(\theta)p)}{eB_3}\right)\mathrm{d}\theta\cdot\nabla_x f = 0.$$

For more details, the reader can refer to [4] where a complete analysis of the coupled Vlasov–Poisson equations (with finite Larmor radius) was performed.

Appendix A

We present here the proofs of Propositions 2.2, 2.3 concerning the convergence of the averages over a flow and the proofs of Propositions 2.4, 2.5 which state the properties of the average operator in the L^1/L^∞ setting.

Proof of Proposition 2.2. We start by checking the uniqueness. Consider two functions $u_1, u_2 \in \ker\mathcal{T}_q$ satisfying

$$\int_{\mathbb{R}^m}\big(u(y)-u_1(y)\big)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^m}\big(u(y)-u_2(y)\big)\varphi(y)\,\mathrm{d}y = 0$$

for any $\varphi \in \ker\mathcal{T}_{q'}$. We deduce that

$$\int_{\mathbb{R}^m}\big(u_1(y)-u_2(y)\big)\varphi(y)\,\mathrm{d}y = 0, \quad \forall\varphi\in\ker\mathcal{T}_{q'}.$$

Taking $\varphi = |u_1-u_2|^{q-2}(u_1-u_2) \in \ker\mathcal{T}_{q'}$ we deduce that $\int_{\mathbb{R}^m}|u_1-u_2|^q\,\mathrm{d}y = 0$ saying that $u_1 = u_2$. In order to justify the existence of $\langle u\rangle$ consider a sequence $(T_n)_n$ such that $\lim_{n\to+\infty} T_n = +\infty$ and $(\langle u\rangle_{T_n})_n$ converges weakly in $L^q(\mathbb{R}^m)$ towards some function $\tilde{u} \in L^q(\mathbb{R}^m)$. Observe that $\tilde{u} \in \ker\mathcal{T}_q$. For this it is sufficient to prove that for any $t\in\mathbb{R}$ and $\psi \in L^{q'}(\mathbb{R}^m)$ we have

$$\int_{\mathbb{R}^m}\tilde{u}(y)\psi\big(Y(-t;y)\big)\,\mathrm{d}y = \int_{\mathbb{R}^m}\tilde{u}(y)\psi(y)\,\mathrm{d}y. \tag{56}$$

Indeed, by using the weak convergence $\lim_{n\to+\infty}\langle u\rangle_{T_n} = \tilde{u}$ we deduce

$$\begin{aligned}
\int_{\mathbb{R}^m} \tilde{u}(y)\psi\big(Y(-t;y)\big)\,\mathrm{d}y &= \lim_{n\to+\infty} \int_{\mathbb{R}^m} \langle u\rangle_{T_n}(y)\psi\big(Y(-t;y)\big)\,\mathrm{d}y \\
&= \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{T_n}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi\big(Y(-t;y)\big)\,\mathrm{d}y\,\mathrm{d}s \\
&= \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{T_n}\int_{\mathbb{R}^m} u\big(Y(s+t;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s \\
&= \lim_{n\to+\infty} \frac{1}{T_n}\int_t^{t+T_n}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s \\
&= \lim_{n\to+\infty} \frac{1}{T_n}\int_{T_n}^{t+T_n}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \lim_{n\to+\infty} \frac{1}{T_n}\int_0^{t}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \lim_{n\to+\infty} \int_{\mathbb{R}^m} \langle u\rangle_{T_n}(y)\psi(y)\,\mathrm{d}y. \qquad (57)
\end{aligned}$$

It is easily seen that

$$\frac{1}{T_n}\left|\int_{T_n}^{t+T_n}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s\right| \leqslant \frac{|t|}{T_n}\|u\|_{L^q(\mathbb{R}^m)}\|\psi\|_{L^{q'}(\mathbb{R}^m)} \qquad (58)$$

and

$$\frac{1}{T_n}\left|\int_0^{t}\int_{\mathbb{R}^m} u\big(Y(s;y)\big)\psi(y)\,\mathrm{d}y\,\mathrm{d}s\right| \leqslant \frac{|t|}{T_n}\|u\|_{L^q(\mathbb{R}^m)}\|\psi\|_{L^{q'}(\mathbb{R}^m)}. \qquad (59)$$

Combining (57)–(59) yields (56), implying that

$$\tilde{u}\big(Y(s;y)\big) = \tilde{u}(y), \quad s \in \mathbb{R},\ \text{a.e. } y \in \mathbb{R}^m.$$

We claim that $\tilde{u}$ satisfies (16). For any $\varphi \in \ker \mathcal{T}_{q'}$ and $s \in \mathbb{R}$ we have $u\varphi \in L^1(\mathbb{R}^m)$ and thus by change of variable along the characteristics we obtain

$$\int_{\mathbb{R}^m} u(y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^m} u\big(Y(s;y)\big)\varphi\big(Y(s;y)\big)\,\mathrm{d}y = \int_{\mathbb{R}^m} u\big(Y(s;y)\big)\varphi(y)\,\mathrm{d}y.$$

Taking the average on $[0, T_n]$ one gets

$$\int_{\mathbb{R}^m} u(y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^m} \left(\frac{1}{T_n}\int_0^{T_n} u\big(Y(s;\cdot)\big)\,\mathrm{d}s\right)(y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}^m} \langle u\rangle_{T_n}(y)\varphi(y)\,\mathrm{d}y.$$

Since $\varphi \in L^{q'}(\mathbb{R}^m)$ we obtain thanks to the weak convergence $\lim_{n\to+\infty}\langle u\rangle_{T_n} = \tilde{u}$ in $L^q(\mathbb{R}^m)$ that

$$\int_{\mathbb{R}^m} \big(u(y) - \tilde{u}(y)\big)\varphi(y)\,\mathrm{d}y = 0, \quad \forall\varphi \in \ker \mathcal{T}_{q'}.$$

Therefore the existence of the element $\langle u\rangle$ in (16) is guaranteed, and by the uniqueness of such element we deduce also the convergence $\lim_{T\to+\infty}\frac{1}{T}\int_0^T u(Y(s;\cdot))\,\mathrm{d}s = \langle u\rangle$ weakly in $L^q(\mathbb{R}^m)$. Similarly one gets

$$\lim_{T\to+\infty}\frac{1}{T}\int_{-T}^{0} u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T} u\big(Y(s;\cdot)\big)\,\mathrm{d}s = \langle u\rangle \quad \text{weakly in } L^q\big(\mathbb{R}^m\big).$$

Since for any $T>0$ we have $\|\langle u\rangle_T\|_{L^q(\mathbb{R}^m)} \leqslant \|u\|_{L^q(\mathbb{R}^m)}$ we deduce that $\|\langle u\rangle\|_{L^q(\mathbb{R}^m)} \leqslant \|u\|_{L^q(\mathbb{R}^m)}$. The linearity of $\langle\cdot\rangle$ follows immediately and we have $\|\langle\cdot\rangle\|_{\mathcal{L}(L^q(\mathbb{R}^m),L^q(\mathbb{R}^m))} \leqslant 1$. $\square$

Proof of Proposition 2.3. We analyze first the case $q=2$. Recall that the adjoint of $\mathcal{T}_2$ satisfies

$$\mathrm{D}\big(\mathcal{T}_2^\star\big) = \mathrm{D}(\mathcal{T}_2), \qquad \mathcal{T}_2^\star u = -\mathcal{T}_2 u, \quad \forall u \in \mathrm{D}(\mathcal{T}_2).$$

Therefore we have $\ker\mathcal{T}_2 = \ker\mathcal{T}_2^\star = (\operatorname{range}\mathcal{T}_2)^\perp$, implying the orthogonal decomposition of $L^2(\mathbb{R}^m)$

$$\ker\mathcal{T}_2 \oplus \overline{\operatorname{range}\mathcal{T}_2} = (\operatorname{range}\mathcal{T}_2)^\perp \oplus \big((\operatorname{range}\mathcal{T}_2)^\perp\big)^\perp = L^2\big(\mathbb{R}^m\big).$$

By Proposition 2.2 we know that for any $u \in L^2(\mathbb{R}^m)$, the function $\langle u\rangle^{(2)}$ is the orthogonal projection of u on the closed subspace $\ker\mathcal{T}_2$ and thus we have the decomposition $u = \langle u\rangle^{(2)} + (u - \langle u\rangle^{(2)})$ with $\langle u\rangle^{(2)} \in \ker\mathcal{T}_2$ and $u - \langle u\rangle^{(2)} \in \overline{\operatorname{range}\mathcal{T}_2}$. As seen before, for any $T>0$ we have

$$\big\langle\langle u\rangle^{(2)}\big\rangle_T = \frac{1}{T}\int_0^T \langle u\rangle^{(2)}\big(Y(s;\cdot)\big)\,\mathrm{d}s = \langle u\rangle^{(2)}$$

and thus

$$\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle^{(2)} + \lim_{T\to+\infty}\big\langle u - \langle u\rangle^{(2)}\big\rangle_T, \quad \text{strongly in } L^2\big(\mathbb{R}^m\big).$$

In order to prove that $\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle^{(2)}$ strongly in $L^2(\mathbb{R}^m)$ it remains to check that $\lim_{T\to+\infty}\langle v\rangle_T = 0$, strongly in $L^2(\mathbb{R}^m)$ for any $v \in \overline{\operatorname{range}\mathcal{T}_2}$. Consider first $v = \mathcal{T}_2 w$ for some $w \in \mathrm{D}_2$. Let us consider a sequence $(w_n)_n \subset C_c^1(\mathbb{R}^m)$ such that

$$\lim_{n\to+\infty}(w_n, \mathcal{T}_2 w_n) = (w, \mathcal{T}_2 w), \quad \text{strongly in } L^2\big(\mathbb{R}^m\big).$$

We have for any $y \in \mathbb{R}^m$

$$\langle \mathcal{T}_2 w_n\rangle_T(y) = \frac{1}{T}\int_0^T (\mathcal{T}_2 w_n)\big(Y(s;y)\big)\,\mathrm{d}s = \frac{1}{T}\int_0^T \frac{d}{ds}\big\{w_n\big(Y(s;y)\big)\big\}\,\mathrm{d}s = \frac{1}{T}\big(w_n\big(Y(T;y)\big) - w_n(y)\big)$$

and therefore

$$\big\|\langle \mathcal{T}_2 w_n\rangle_T\big\|_{L^2(\mathbb{R}^m)} \leqslant \frac{2}{T}\|w_n\|_{L^2(\mathbb{R}^m)}.$$

Passing to the limit for $n \to +\infty$ one gets $\|\langle v\rangle_T\|_{L^2(\mathbb{R}^m)} \leqslant \frac{2}{T}\|w\|_{L^2(\mathbb{R}^m)}$, implying that $\lim_{T\to+\infty}\langle v\rangle_T = 0$ strongly in $L^2(\mathbb{R}^m)$. Consider now a function $v \in \overline{\operatorname{range}\mathcal{T}_2}$. For any $\delta > 0$ there exists $v_\delta \in \operatorname{range}\mathcal{T}_2$ such that $\|v - v_\delta\|_{L^2(\mathbb{R}^m)} < \delta$. We have

$$\begin{aligned}\big\|\langle v\rangle_T\big\|_{L^2(\mathbb{R}^m)} &\leqslant \big\|\langle v - v_\delta\rangle_T\big\|_{L^2(\mathbb{R}^m)} + \big\|\langle v_\delta\rangle_T\big\|_{L^2(\mathbb{R}^m)}\\ &\leqslant \|v - v_\delta\|_{L^2(\mathbb{R}^m)} + \big\|\langle v_\delta\rangle_T\big\|_{L^2(\mathbb{R}^m)}\\ &\leqslant \delta + \big\|\langle v_\delta\rangle_T\big\|_{L^2(\mathbb{R}^m)}.\end{aligned}$$

Passing to the limit for $T \to +\infty$ we obtain

$$\limsup_{T\to+\infty}\big\|\langle v\rangle_T\big\|_{L^2(\mathbb{R}^m)} \leqslant \delta, \quad \forall\delta > 0,$$

and consequently $\lim_{T\to+\infty}\|\langle v\rangle_T\|_{L^2(\mathbb{R}^m)} = 0$ for any $v \in \overline{\operatorname{range}\mathcal{T}_2}$.

Consider now the general case $q \in (1,+\infty)$. By density arguments it is sufficient to treat the case of functions $u \in C_c(\mathbb{R}^m)$. Since $C_c(\mathbb{R}^m) \subset L^r(\mathbb{R}^m)$ for any $r \in (1,+\infty)$ we deduce thanks to Corollary 2.3 that $\langle u\rangle \in L^r(\mathbb{R}^m)$ and $\|\langle u\rangle\|_{L^r(\mathbb{R}^m)} \leqslant \|u\|_{L^r(\mathbb{R}^m)}$ for any $r \in (1,+\infty)$. By the previous step we know that $\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle$ strongly in $L^2(\mathbb{R}^m)$ and it is easily seen that $\langle u\rangle \in L^1(\mathbb{R}^m)\cap L^\infty(\mathbb{R}^m)$ and satisfies $\|\langle u\rangle\|_{L^1(\mathbb{R}^m)} \leqslant \|u\|_{L^1(\mathbb{R}^m)}$, $\|\langle u\rangle\|_{L^\infty(\mathbb{R}^m)} \leqslant \|u\|_{L^\infty(\mathbb{R}^m)}$ (use for example the convergence $\lim_{T\to+\infty}\langle u\rangle_T = \langle u\rangle$ in $\mathcal{D}'(\mathbb{R}^m)$ and the bounds $\|\langle u\rangle_T\|_{L^1(\mathbb{R}^m)} \leqslant \|u\|_{L^1(\mathbb{R}^m)}$, $\|\langle u\rangle_T\|_{L^\infty(\mathbb{R}^m)} \leqslant \|u\|_{L^\infty(\mathbb{R}^m)}$ for any $T > 0$). If $q \in (1,2)$ we have by interpolation inequalities

$$\begin{aligned}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^q(\mathbb{R}^m)} &\leqslant \big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^1(\mathbb{R}^m)}^{\frac{2}{q}-1}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^2(\mathbb{R}^m)}^{2-\frac{2}{q}}\\ &\leqslant \big(2\|u\|_{L^1(\mathbb{R}^m)}\big)^{\frac{2}{q}-1}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^2(\mathbb{R}^m)}^{2-\frac{2}{q}} \to 0 \quad \text{as } T \to +\infty.\end{aligned}$$

If $q \in (2,+\infty)$ we have

$$\begin{aligned}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^q(\mathbb{R}^m)} &\leqslant \big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^2(\mathbb{R}^m)}^{\frac{2}{q}}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^\infty(\mathbb{R}^m)}^{1-\frac{2}{q}}\\ &\leqslant \big(2\|u\|_{L^\infty(\mathbb{R}^m)}\big)^{1-\frac{2}{q}}\big\|\langle u\rangle_T - \langle u\rangle\big\|_{L^2(\mathbb{R}^m)}^{\frac{2}{q}} \to 0 \quad \text{as } T \to +\infty. \qquad \square\end{aligned}$$

Proof of Proposition 2.4. Consider a sequence $(u_n)_n \subset C_c(\mathbb{R}^m)$ satisfying $\lim_{n\to+\infty} u_n = u$ in $L^1(\mathbb{R}^m)$. For any $n \in \mathbb{N}$, $q \in (1,+\infty)$ the function u_n belongs to $L^q(\mathbb{R}^m)$ and by Proposition 2.2 and Corollary 2.3 we know that there is $\langle u_n\rangle \in \ker\mathcal{T}_q$, $\forall q \in (1,+\infty)$, satisfying

$$\int_{\mathbb{R}^m}\big(u_n(y) - \langle u_n\rangle(y)\big)\varphi(y)\,\mathrm{d}y = 0, \quad \forall\varphi \in \ker\mathcal{T}_{q'},\ q \in (1,+\infty). \tag{60}$$

In particular since $\|\langle\cdot\rangle\|_{\mathcal{L}(L^q(\mathbb{R}^m),L^q(\mathbb{R}^m))} \leqslant 1$ we have

$$\int_{\mathbb{R}^m} \big|\langle u_n\rangle - \langle u_l\rangle\big|^q \,\mathrm{d}y \leqslant \int_{\mathbb{R}^m} |u_n - u_l|^q \,\mathrm{d}y, \quad n, l \in \mathbb{N}. \tag{61}$$

By Fatou's lemma we deduce that

$$\int_{\mathbb{R}^m} \big|\langle u_n\rangle - \langle u_l\rangle\big| \,\mathrm{d}y \leqslant \liminf_{q \searrow 1} \int_{\mathbb{R}^m} \big|\langle u_n\rangle - \langle u_l\rangle\big|^q \,\mathrm{d}y$$

and by dominated convergence theorem we have

$$\lim_{q \searrow 1} \int_{\mathbb{R}^m} |u_n - u_l|^q \,\mathrm{d}y = \int_{\mathbb{R}^m} |u_n - u_l| \,\mathrm{d}y.$$

Therefore, passing to the limit for $q \searrow 1$ in (61) yields

$$\int_{\mathbb{R}^m} \big|\langle u_n\rangle - \langle u_l\rangle\big| \,\mathrm{d}y \leqslant \int_{\mathbb{R}^m} |u_n - u_l| \,\mathrm{d}y$$

saying that $(\langle u_n\rangle)_n$ is a Cauchy sequence in $L^1(\mathbb{R}^m)$. Let us denote by $\langle u\rangle$ the limit of $(\langle u_n\rangle)_n$ in $L^1(\mathbb{R}^m)$. Since $(\langle u_n\rangle)_n$ are constant along the flow we check easily that $\langle u\rangle$ is also constant along the flow. Moreover, $\langle u\rangle$ belongs to $L^1(\mathbb{R}^m)$ and by the construction of $\mathcal{O}$ we deduce that $\langle u\rangle = 0$ on $\mathcal{O}$. Consider a function $\varphi \in \ker \mathcal{T}_\infty$. Applying (60) with

$$\big(\xi^{1/q} + \big|\langle u_n\rangle\big|\big)^{q-1} \varphi \mathbf{1}_{\mathbb{R}^m \setminus \mathcal{O}} \in \ker \mathcal{T}_{q'}$$

(where $\xi(\cdot)$ is the function appearing in (18)) we deduce that

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} u_n \big(\xi^{1/q} + \big|\langle u_n\rangle\big|\big)^{q-1} \varphi \,\mathrm{d}y = \int_{\mathbb{R}^m \setminus \mathcal{O}} \langle u_n\rangle \big(\xi^{1/q} + \big|\langle u_n\rangle\big|\big)^{q-1} \varphi \,\mathrm{d}y. \tag{62}$$

We keep n fixed for the moment and we intend to pass to the limit for $q \searrow 1$ in the above equality. We use the trivial inequality $x^z \leqslant 1 + x$, for any $x > 0$, $z \in (0,1)$. One gets for any $q \in (1,2)$

$$\big(\big(\xi(y)\big)^{1/q} + \big|\langle u_n\rangle(y)\big|\big)^{q-1} \leqslant 1 + \big(\xi(y)\big)^{1/q} + \big|\langle u_n\rangle(y)\big| \leqslant 2 + \xi(y) + \big|\langle u_n\rangle(y)\big|$$

and thus

$$\begin{aligned} \big|u_n(y)\big(\big(\xi(y)\big)^{1/q} + \big|\langle u_n\rangle(y)\big|\big)^{q-1}\varphi(y)\big| &\leqslant \|\varphi\|_{L^\infty(\mathbb{R}^m)} \|u_n\|_{L^\infty(\mathbb{R}^m)} \big(\xi(y) + \big|\langle u_n\rangle(y)\big|\big) \\ &\quad + 2\|\varphi\|_{L^\infty(\mathbb{R}^m)} \big|u_n(y)\big| \in L^1\big(\mathbb{R}^m \setminus \mathcal{O}\big). \end{aligned}$$

Since $\xi > 0$ on $\mathbb{R}^m \setminus \mathcal{O}$ we have the pointwise convergence

$$\lim_{q \searrow 1} u_n(y)\big(\big(\xi(y)\big)^{1/q} + \big|\langle u_n\rangle(y)\big|\big)^{q-1}\varphi(y) = u_n(y)\varphi(y), \quad y \in \mathbb{R}^m \setminus \mathcal{O},$$

and thus we deduce by Lebesgue's theorem

$$\lim_{q\searrow 1}\int_{\mathbb{R}^m\setminus\mathcal{O}} u_n(y)\big(\big(\xi(y)\big)^{1/q}+\big|\langle u_n\rangle(y)\big|\big)^{q-1}\varphi(y)\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}} u_n(y)\varphi(y)\,\mathrm{d}y. \tag{63}$$

By similar arguments we can pass to the limit for $q\searrow 1$ in the right-hand side of (62) (for this observe also that, by Corollary 2.1, we have $\|\langle u_n\rangle\|_{L^\infty(\mathbb{R}^m)}\leqslant\|u_n\|_{L^\infty(\mathbb{R}^m)}$)

$$\lim_{q\searrow 1}\int_{\mathbb{R}^m\setminus\mathcal{O}} \langle u_n\rangle(y)\big(\big(\xi(y)\big)^{1/q}+\big|\langle u_n\rangle(y)\big|\big)^{q-1}\varphi(y)\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}} \langle u_n\rangle(y)\varphi(y)\,\mathrm{d}y. \tag{64}$$

Combining (62)–(64) yields

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}\big(u_n(y)-\langle u_n\rangle(y)\big)\varphi(y)\,\mathrm{d}y=0,\quad \forall\varphi\in\ker\mathcal{T}_\infty.$$

Passing now to the limit for $n\to+\infty$ implies

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}\big(u(y)-\langle u\rangle(y)\big)\varphi(y)\,\mathrm{d}y=0,\quad \forall\varphi\in\ker\mathcal{T}_\infty. \tag{65}$$

We consider the function $\varphi=\operatorname{sgn}\langle u\rangle$. Since $\langle u\rangle$ is constant along the flow, we have $\varphi\in\ker\mathcal{T}_\infty$ and therefore we deduce thanks to (65)

$$\int_{\mathbb{R}^m}\big|\langle u\rangle\big|\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}}\big|\langle u\rangle\big|\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}} u\operatorname{sgn}\langle u\rangle\,\mathrm{d}y\leqslant\int_{\mathbb{R}^m\setminus\mathcal{O}}|u|\,\mathrm{d}y\leqslant\int_{\mathbb{R}^m}|u|\,\mathrm{d}y.$$

The uniqueness of the function $\langle u\rangle$ constructed above is immediate. Indeed, let us consider two functions $u_1,u_2\in\ker\mathcal{T}_1$ satisfying

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}(u-u_1)\varphi\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}}(u-u_2)\varphi\,\mathrm{d}y=0,\quad \forall\varphi\in\ker\mathcal{T}_\infty.$$

By the definition of $\mathcal{O}$ we have $u_1=u_2=0$ on $\mathcal{O}$ and taking $\varphi=\operatorname{sgn}(u_1-u_2)\in\ker\mathcal{T}_\infty$ we deduce

$$\int_{\mathbb{R}^m\setminus\mathcal{O}}|u_1-u_2|\,\mathrm{d}y=\int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1-u_2)\varphi\,\mathrm{d}y=0.$$

Finally $u_1=u_2$ on $\mathbb{R}^m$. The linearity of the application $u\in L^1(\mathbb{R}^m)\to\langle u\rangle\in L^1(\mathbb{R}^m)$ follows easily by using the characterization (19). $\square$

Proof of Proposition 2.5. In order to prove the uniqueness, consider $u_1,u_2\in\ker\mathcal{T}_\infty$ satisfying $u_1=u_2=0$ on $\mathcal{O}$ and $\int_{\mathbb{R}^m\setminus\mathcal{O}}(u_1-u_2)\varphi\,\mathrm{d}y=0$ for any $\varphi\in\ker\mathcal{T}_1$. By Proposition 2.4 we know that for any

$\psi \in L^1(\mathbb{R}^m)$ there is $\langle \psi \rangle \in \ker \mathcal{T}_1$ such that

$$\int_{\mathbb{R}^m \setminus \mathcal{O}} \big(\psi - \langle \psi \rangle\big) v \, \mathrm{d}y = 0, \quad \forall v \in \ker \mathcal{T}_\infty.$$

In particular we have for $v = u_1 - u_2 \in \ker \mathcal{T}_\infty$

$$\int_{\mathbb{R}^m} (u_1 - u_2)\psi \, \mathrm{d}y = \int_{\mathbb{R}^m \setminus \mathcal{O}} (u_1 - u_2)\psi \, \mathrm{d}y = \int_{\mathbb{R}^m \setminus \mathcal{O}} (u_1 - u_2)\langle \psi \rangle \, \mathrm{d}y = 0, \quad \forall \psi \in L^1\big(\mathbb{R}^m\big),$$

implying that $u_1 = u_2$. The existence follows by considering $(T_n)_n$ such that $\lim_{n \to +\infty} T_n = +\infty$ and

$$\langle u \rangle_{T_n} \rightharpoonup \tilde{u} \quad \text{weakly} \star \text{ in } L^\infty\big(\mathbb{R}^m \setminus \mathcal{O}\big)$$

for some function $\tilde{u} \in L^\infty(\mathbb{R}^m \setminus \mathcal{O})$. As in the proof of Proposition 2.2 we check that

$$\tilde{u} \in \ker \mathcal{T}_\infty, \quad \int_{\mathbb{R}^m \setminus \mathcal{O}} (u - \tilde{u})\varphi \, \mathrm{d}y = 0, \quad \forall \varphi \in \ker \mathcal{T}_1, \quad \|\tilde{u}\|_{L^\infty(\mathbb{R}^m \setminus \mathcal{O})} \leqslant \|u\|_{L^\infty(\mathbb{R}^m \setminus \mathcal{O})}.$$

We take $\langle u \rangle = \tilde{u} \mathbf{1}_{\mathbb{R}^m \setminus \mathcal{O}}$. $\square$

Appendix B

This section contains the proof of the regularity result stated in Section 3.

Proof of Proposition 3.2. For any $\varepsilon > 0$ let u^ε be the solution of (30). We intend to estimate $\|\mathcal{T}_q u^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^r \|\mathcal{T}_q^i u^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))}$ and $\|\partial_t u^\varepsilon\|_{L^1([0,T];L^q(\mathbb{R}^m))}$ uniformly with respect to $\varepsilon > 0$. Consider the sequences of smooth functions $(u_{0n})_n$, $(f_n)_n$ such that

$$\lim_{n \to +\infty} u_{0n} = u_0, \qquad \lim_{n \to +\infty} \mathcal{T}_q^i u_{0n} = \mathcal{T}_q^i u_0, \quad i \in \{0, 1, \ldots, r\} \text{ in } L^q\big(\mathbb{R}^m\big),$$
$$\lim_{n \to +\infty} f_n = f, \qquad \lim_{n \to +\infty} \mathcal{T}_q^i f_n = \mathcal{T}_q^i f, \quad i \in \{0, 1, \ldots, r\} \text{ in } L^1\big([0, T]; L^q\big(\mathbb{R}^m\big)\big)$$

and let us denote by $(u_n^\varepsilon)_n$ the solutions of (30) corresponding to the initial conditions $(u_{0n})_n$ and the source terms $(f_n)_n$. Actually $(u_n^\varepsilon)_n$ are strong solutions. It is easily seen that

$$\big\|u_n^\varepsilon(t) - u^\varepsilon(t)\big\|_{L^q(\mathbb{R}^m)} \leqslant \|u_{0n} - u_0\|_{L^q(\mathbb{R}^m)} + \int_0^t \big\|f_n(s) - f(s)\big\|_{L^q(\mathbb{R}^m)} \, \mathrm{d}s, \quad t \in [0, T],$$

and therefore $\lim_{n \to +\infty} u_n^\varepsilon = u^\varepsilon$ in $L^\infty([0, T]; L^q(\mathbb{R}^m))$. Assume for the moment that ε, n are fixed and let us estimate $\sum_{i=0}^r \|\mathcal{T}_q^i u_n^\varepsilon\|_{L^\infty([0,T];L^q(\mathbb{R}^m))}$ and $\|\partial_t u_n^\varepsilon\|_{L^1([0,T];L^q(\mathbb{R}^m))}$. Take $h \in \mathbb{R}$, $i \in \{1, \ldots, r\}$, and consider the functions

$$u^\varepsilon_{nh}(t,y) = u^\varepsilon_n\big(t, Y^i(h;y)\big), \qquad a_h(t,y) = \frac{\partial Y^i}{\partial y}\big(-h; Y^i(h;y)\big)a\big(t, Y^i(h;y)\big),$$
$$b_h(y) = \frac{\partial Y^i}{\partial y}\big(-h; Y^i(h;y)\big)b\big(Y^i(h;y)\big), \qquad u_{0nh}(y) = u_{0n}\big(Y^i(h;y)\big),$$
$$f_{nh}(t,y) = f_n\big(t, Y^i(h;y)\big).$$

A direct computation shows that

$$\begin{cases} \partial_t u^\varepsilon_{nh} + a_h(t,y)\cdot\nabla_y u^\varepsilon_{nh} + \dfrac{b_h(y)}{\varepsilon}\cdot\nabla_y u^\varepsilon_{nh} = f_{nh}(t,y), & (t,y)\in(0,T)\times\mathbb{R}^m,\\ u^\varepsilon_{nh}(0,y) = u_{0nh}(y), & y\in\mathbb{R}^m. \end{cases} \tag{66}$$

Combining with the formulation (30) of u^ε_n one gets

$$\begin{cases} \partial_t\left(\dfrac{u^\varepsilon_{nh}-u^\varepsilon_n}{h}\right) + \dfrac{a_h - a}{h}\cdot\nabla_y u^\varepsilon_{nh} + a(t,y)\cdot\nabla_y\left(\dfrac{u^\varepsilon_{nh}-u^\varepsilon_n}{h}\right) & \\ \quad + \dfrac{b_h - b}{\varepsilon h}\cdot\nabla_y u^\varepsilon_{nh} + \dfrac{b(y)}{\varepsilon}\cdot\nabla_y\left(\dfrac{u^\varepsilon_{nh}-u^\varepsilon_n}{h}\right) = \dfrac{f_{nh}-f_n}{h}, & (t,y)\in(0,T)\times\mathbb{R}^m,\\ \dfrac{u^\varepsilon_{nh}(0,y)-u^\varepsilon_n(0,y)}{h} = \dfrac{u_{0nh}(y)-u_{0n}(y)}{h}, & y\in\mathbb{R}^m. \end{cases} \tag{67}$$

Obviously we have

$$\lim_{h\to 0}\frac{u^\varepsilon_{nh}-u^\varepsilon_n}{h} = \lim_{h\to 0}\frac{u^\varepsilon_n(t,Y^i(h;y)) - u^\varepsilon_n(t,y)}{h} = b^i(y)\cdot\nabla_y u^\varepsilon_n(t,y) = \mathcal{T}^i_q u^\varepsilon_n,$$
$$\lim_{h\to 0}\frac{f_{nh}-f_n}{h} = \lim_{h\to 0}\frac{f_n(t,Y^i(h;y)) - f_n(t,y)}{h} = b^i(y)\cdot\nabla_y f_n(t,y) = \mathcal{T}^i_q f_n,$$
$$\lim_{h\to 0}\frac{u_{0nh}-u_{0n}}{h} = \lim_{h\to 0}\frac{u_{0n}(Y^i(h;y)) - u_{0n}(y)}{h} = b^i(y)\cdot\nabla_y u_{0n}(y) = \mathcal{T}^i_q u_{0n}.$$

Taking the derivatives with respect to y and then with respect to h in the equality $Y^i(-h; Y^i(h; y)) = y$, we deduce after some easy manipulations that

$$\lim_{h\to 0}\frac{1}{h}\left\{\frac{\partial Y^i}{\partial y}\big(-h; Y^i(h;y)\big) - I_m\right\} = -\frac{\partial b^i}{\partial y}(y).$$

By direct computations we obtain immediately

$$\lim_{h\to 0}\frac{a_h - a}{h} = \big(b^i\cdot\nabla_y\big)a - (a\cdot\nabla_y)b^i = \big[b^i, a\big],$$
$$\lim_{h\to 0}\frac{b_h - b}{h} = \big(b^i\cdot\nabla_y\big)b - (b\cdot\nabla_y)b^i = \big[b^i, b\big] = 0.$$

By passing to the limit for $h\to 0$ in (67) we deduce that $\mathcal{T}^i_q u^\varepsilon_n$ solves weakly the problem

$$\begin{cases} \partial_t\big(\mathcal{T}^i_q u^\varepsilon_n\big) + a\cdot\nabla_y\big(\mathcal{T}^i_q u^\varepsilon_n\big) + \dfrac{b}{\varepsilon}\cdot\nabla_y\big(\mathcal{T}^i_q u^\varepsilon_n\big) = \mathcal{T}^i_q f_n - \big[b^i, a\big]\cdot\nabla_y u^\varepsilon_n,\\ \mathcal{T}^i_q u^\varepsilon_n(0,\cdot) = \mathcal{T}^i_q u_{0n}. \end{cases} \tag{68}$$

As in the proof of Proposition 3.1 we obtain for any $t \in [0, T]$ and $i \in \{1, \dots, r\}$

$$\left\|\mathcal{T}_q^i u_n^\varepsilon(t)\right\|_{L^q(\mathbb{R}^m)} \leqslant \left\|\mathcal{T}_q^i u_{0n}\right\|_{L^q(\mathbb{R}^m)} + \int_0^t \left\|\mathcal{T}_q^i f_n(s) - \left[b^i, a(s)\right] \cdot \nabla_y u_n^\varepsilon(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s. \tag{69}$$

Since $a = \sum_{k=0}^r \alpha_k b^k$ we obtain by direct computation, with the notation $\mathcal{T}_q^0 := \mathcal{T}_q$

$$\left[b^i, a\right] = \sum_{k=0}^r \left(\mathcal{T}_q^i \alpha_k\right) b^k$$

and therefore

$$\left[b^i, a\right] \cdot \nabla_y u_n^\varepsilon = \sum_{k=0}^r \left(\mathcal{T}_q^i \alpha_k\right)\left(\mathcal{T}_q^k u_n^\varepsilon\right).$$

Consequently (69) implies

$$\begin{aligned}\left\|\mathcal{T}_q^i u_n^\varepsilon(t)\right\|_{L^q(\mathbb{R}^m)} \leqslant{} & \left\|\mathcal{T}_q^i u_{0n}\right\|_{L^q(\mathbb{R}^m)} + \int_0^t \left\|\mathcal{T}_q^i f_n(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s \\ & + \int_0^t \sum_{k=0}^r \left\|b^i \cdot \nabla_y \alpha_k(s)\right\|_{L^\infty(\mathbb{R}^m)} \left\|\mathcal{T}_q^k u_n^\varepsilon(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s.\end{aligned} \tag{70}$$

Actually (70) holds also for b^i replaced by $b^0 = b$ since $[b, b] = 0$

$$\begin{aligned}\left\|\mathcal{T}_q^0 u_n^\varepsilon(t)\right\|_{L^q(\mathbb{R}^m)} \leqslant{} & \left\|\mathcal{T}_q^0 u_{0n}\right\|_{L^q(\mathbb{R}^m)} + \int_0^t \left\|\mathcal{T}_q^0 f_n(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s \\ & + \int_0^t \sum_{k=0}^r \left\|b^0 \cdot \nabla_y \alpha_k(s)\right\|_{L^\infty(\mathbb{R}^m)} \left\|\mathcal{T}_q^k u_n^\varepsilon(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s.\end{aligned} \tag{71}$$

Summing up the above inequalities one gets

$$\begin{aligned}\sum_{i=0}^r \left\|\mathcal{T}_q^i u_n^\varepsilon(t)\right\|_{L^q(\mathbb{R}^m)} \leqslant{} & \sum_{i=0}^r \left\|\mathcal{T}_q^i u_{0n}\right\|_{L^q(\mathbb{R}^m)} + \int_0^t \sum_{i=0}^r \left\|\mathcal{T}_q^i f_n(s)\right\|_{L^q(\mathbb{R}^m)} \mathrm{d}s \\ & + \sum_{i=0}^r \sum_{k=0}^r \int_0^t \left\|b^i \cdot \nabla_y \alpha_k(s)\right\|_{L^\infty(\mathbb{R}^m)} \left\|\mathcal{T}_q^k u_n^\varepsilon(s)\right\|_{L^q(\mathbb{R}^m)}.\end{aligned} \tag{72}$$

By Gronwall's lemma we deduce that for any $t \in [0, T]$

$$\sum_{i=0}^{r} \left\| \mathcal{T}_q^i u_n^\varepsilon \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leqslant C \sum_{i=0}^{r} \left\{ \left\| \mathcal{T}_q^i u_{0n} \right\|_{L^q(\mathbb{R}^m)} + \left\| \mathcal{T}_q^i f_n \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} \right\} \tag{73}$$

for some constant depending on $\sum_{0 \leqslant i,j \leqslant r} \|b^i \cdot \nabla_y \alpha_j\|_{L^1([0,T];L^\infty(\mathbb{R}^m))}$. After extraction eventually we can assume that $(\mathcal{T}_q^i u_n^\varepsilon)_n$ converges weakly $\star$ in $L^\infty([0,T];L^q(\mathbb{R}^m))$ towards some function $w^i \in L^\infty([0,T];L^q(\mathbb{R}^m))$ for any $i \in \{0,1,\dots,r\}$. Since we know that $\lim_{n\to+\infty} u_n^\varepsilon = u^\varepsilon$ in $L^\infty([0,T];L^q(\mathbb{R}^m))$ it is easily seen that

$$u^\varepsilon(t) \in \bigcap_{i=0}^{r} \mathrm{D}\big(\mathcal{T}_q^i\big), \qquad \mathcal{T}_q^i u^\varepsilon(t) = w^i(t), \quad t \in [0,T].$$

Moreover, passing to the limit with respect to n in (73) and taking into account that $\lim_{n\to+\infty} \mathcal{T}_q u_{0n} = \mathcal{T}_q u_0 = 0$ in $L^q(\mathbb{R}^m)$ and $\lim_{n\to+\infty} \mathcal{T}_q f_n = \mathcal{T}_q f = 0$ in $L^1([0,T];L^q(\mathbb{R}^m))$ we obtain

$$\sum_{i=1}^{r} \left\| \mathcal{T}_q^i u^\varepsilon \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leqslant C \sum_{i=1}^{r} \left\{ \left\| \mathcal{T}_q^i u_0 \right\|_{L^q(\mathbb{R}^m)} + \left\| \mathcal{T}_q^i f \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} \right\}. \tag{74}$$

Recall that the weak solution u constructed in Proposition 3.1 has been obtained by taking a weak $\star$ limit point of the family $(u^\varepsilon)_{\varepsilon>0}$ in $L^\infty([0,T];L^q(\mathbb{R}^m))$. Therefore we deduce by passing to the limit for $\varepsilon \searrow 0$ in (74) that $u(t) \in \bigcap_{i=1}^r \mathrm{D}(\mathcal{T}_q^i)$, $t \in [0,T]$, and

$$\sum_{i=1}^{r} \left\| \mathcal{T}_q^i u \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \leqslant C \sum_{i=1}^{r} \left\{ \left\| \mathcal{T}_q^i u_0 \right\|_{L^q(\mathbb{R}^m)} + \left\| \mathcal{T}_q^i f \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} \right\}. \tag{75}$$

Since $\mathcal{T}_q u = 0$, observe also that

$$\left\| a(t) \cdot \nabla_y u(t) \right\|_{L^q(\mathbb{R}^m)} = \left\| \sum_{i=1}^{r} \alpha_i(t) b^i \cdot \nabla_y u(t) \right\|_{L^q(\mathbb{R}^m)} \leqslant \sum_{i=1}^{r} \left\| \alpha_i(t) \right\|_{L^\infty(\mathbb{R}^m)} \left\| \mathcal{T}_q^i u(t) \right\|_{L^q(\mathbb{R}^m)}$$

and thus

$$\begin{aligned}
\|\partial_t u\|_{L^1([0,T];L^q(\mathbb{R}^m))} &= \left\| f - \langle a \cdot \nabla_y u \rangle^{(q)} \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} \\
&\leqslant \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^{r} \left\| \mathcal{T}_q^i u \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \|\alpha_i\|_{L^1([0,T];L^\infty(\mathbb{R}^m))} \\
&\leqslant \|f\|_{L^1([0,T];L^q(\mathbb{R}^m))} + C \sum_{i=1}^{r} \left\{ \left\| \mathcal{T}_q^i f \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \left\| \mathcal{T}_q^i u_0 \right\|_{L^q(\mathbb{R}^m)} \right\}.
\end{aligned}$$

When f belongs to $L^\infty([0,T];L^q(\mathbb{R}^m))$ and $\alpha_i \in L^\infty([0,T];L^\infty(\mathbb{R}^m))$ for any $i \in \{1,\dots,r\}$ we obtain

$$\begin{aligned}
\|\partial_t u\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} &\leqslant \|f\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + \sum_{i=1}^{r} \|\alpha_i\|_{L^\infty([0,T];L^\infty(\mathbb{R}^m))} \left\| \mathcal{T}_q^i u \right\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} \\
&\leqslant \|f\|_{L^\infty([0,T];L^q(\mathbb{R}^m))} + C \sum_{i=1}^{r} \left\{ \left\| \mathcal{T}_q^i f \right\|_{L^1([0,T];L^q(\mathbb{R}^m))} + \left\| \mathcal{T}_q^i u_0 \right\|_{L^q(\mathbb{R}^m)} \right\}. \qquad \square
\end{aligned}$$

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